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NOWHERE-ZERO 30-FLOW ON BIDIRECTED GRAPHS

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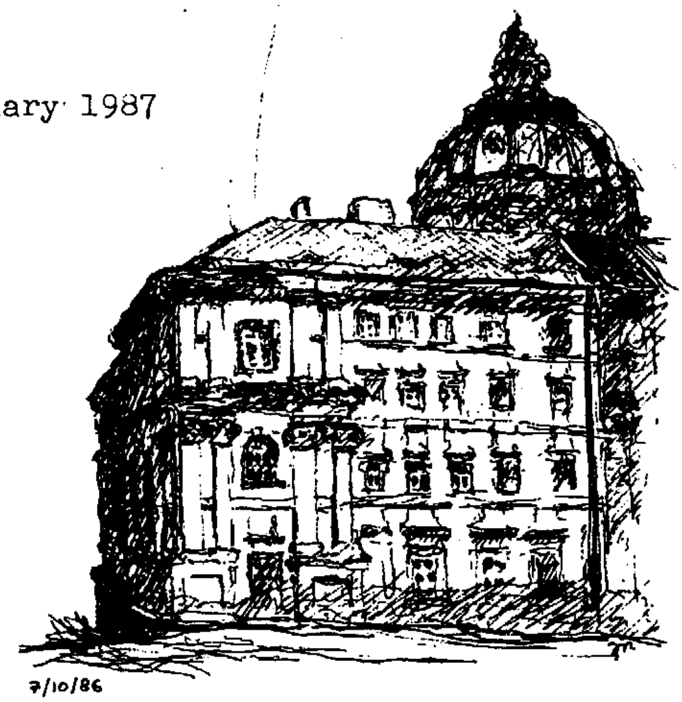

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Nowhere-zero 30-flow on bidirected graphs

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This article concerns Bouchet's conjecture about flows on bidirected graphs. We prove that every graph without signed graphic isthmus can be provided with a nowhere-zero integral flow with absolute values less then 30. This approaches Bouchet's 6-flow conjecture and improves his 216-flow theorem.

I. Introduction

Graphs will be finit and undirected, loops and multiple edges are possible. For convenience we will consider that each edge is constituted of two distinct half-edges each having a single endpoint, so the endpoints of the edge consist of the endpoints of them. The set of all half-edges. edges and vertices of a graph G will be denoted H(G), E(G) and V(G), respectively. For v from V(G), H(v) will be the set of all half-edges incident to v. For a half-edge h we will denote e_h the edge containing it.

A bidirected graph is a graph together with a signature of its edges, $\sigma:E(G)\rightarrow\{+1,-1\}$. A flow on a bidirected graph $G=(V,E,\sigma)$ is a mapping $\phi:E\rightarrow Z$ such that:

i) $\sum_{h\in H(v)} \phi(h) = 0$ for all $v\in V$,

ii) $\phi(h) = -\sigma(e_h).\phi(h')$ for all distinct h,h' such that $e_h = e_{h'}$.

We define $\phi(e_h) = |\phi(h)|$. A nowhere-zero flow or a k-flow is a flow satisfying $\phi(e) \neq 0$ or $\phi(e)<k$ for all $e\in E(G)$, respectively. The support of a flow ϕ on G is the set $S(\phi) = \{e\in E(G) \mid \phi(e) \neq 0\}$.

We can define following operations on bidirected graphs: Switching G at a vertex v ($G\times v$) means changing signs of edges incident to v.

Contraction of a positive edge e (G.e) is deleting e and identifying its endpoints.

Division of an edge e (G+e) with endpoints u,v is deleting e and adding a new wertex w and two new edges e' and e'', e' incident to v and w and e'' incident to w and u, with signs $\sigma(e') = \sigma(e)$, $\sigma(e'') = +1$.

<u>1.1 Note:</u> It is easy to see that for a flow ϕ on G:

i) The mapping ϕ' defined by $\phi'(h) = -\phi(h)$ for $h \in H(v)$ and $\phi'(h) = \phi(h)$ in the other cases is a flow on G×v.

ii) Restriction of ϕ to $E(G)-\{e\}$ is a flow on G.e .

iii) There exists a valuation of half-edges of e',e'' such that $\phi'(e') = \phi'(e'') = \phi(e)$ and this valuation together with restriction of ϕ to $E(G)-\{e\}$ is a flow on G+e .

Moreover, equality $\phi'(e) = \phi(e)$ for all edges assistant at no operation hold and so if ϕ is a nowhere-zero k-flow than ϕ' from case i), ii) or iii) is a nowhere-zero k-flow, too.

A cycle is 2-regular connected subgraph, a path is connected subgraph with exactly two vertices of degree one and the others of degree two. A cycle and a path will be often identified with their edge-sets. The sign of a cycle or a path is the product of its edge-signs. A cycle is balanced or umbalanced if its sign is positive or negative, respectively. A bidirected graph is balanced if all its cycles are balanced, otherwise it is unbalanced. G is almost balanced if it has not two edge-disjoint unbalanced cycles, G is 3-unbalanced if for every connected and balanced subgraph G', the set of half-edges not in H(G') and incident to a vertex of V(G'), has at least 3 elements. Several examples of unbalanced almost balanced graphs are given on fig.1.

An elementary support is either balanced cycle or two vertex-disjoint unbalanced cycles together with a simple connecting path P meeting the cycles at its endpoints. The path P can be empty if the cycles have only one common vertex.

Bouchet [1] proved follouing useful lemmas:

1.2 Lemma: [Proposition 3.2, 3.3] Every flow ϕ on G is a sum of principal flows. These are flows which supports are elementary supports with valuation 1 on all cycles and 2 on the connecting path.

A signed graphic isthmus is such an edge which is in no elementary support.

1.3 Lemma: [Proposition 3.1] There exists a nowhere-zero flow on G iff G has no signed graphic isthmus.

1.4 Lemma: [Proposition 4.2 by a little discusion]
Let k>2, if G is an unbalanced or almost balanced umbalanced graph without signed graphic isthmus with minimum number of edges which can not be provided with a nowhere-zero k-flow, then G is 3-umbalanced.

1.5 Lemma: [Proposition 3.5] Let ϕ is a flow on G, $k > 1$. Then exists a flow ϕ' on G satisfying:

i) $\phi'(h) \equiv \phi(h) \mod k$ for all $h \in H(G)$,

ii) $|\phi(h)| < 2.k$ for all $h \in H(G)$.

Moreover, if G is almost balanced, then $|\phi(h)| < k$ for all $h \in H(G)$.

II. Closure operator

The idea of using a closure operator to prove the existence of a flow is due to Seymour [3].

For an integer k>1, we define k-closure of $X \subseteq E(G)$, $\langle X \rangle_k$, as follows: $\langle X \rangle_k$ is the smallest set $Y \subseteq E$ satisfying:

i) $X \subseteq Y$

ii) for every elementary support S either $S \subseteq Y$ or $|S-Y| > k$.

It is easy to see that if both Y_1 and Y_2 satisfy i) and ii), then so does $Y_1 \cap Y_2$, and so the definition of $\langle X \rangle_k$ is correct. Also for all $X, Y \subseteq E(G)$, the relations $X \subseteq \langle X \rangle_k$, $\langle \langle X \rangle_k \rangle_k = \langle X \rangle_k$ and $X \subseteq Y \Rightarrow \langle X \rangle_k \subseteq \langle Y \rangle_k$ hold.

<u>2.1 Lemma:</u> Let $G=(V,E,\sigma)$ be a bidirected graph and $k>2$ be an odd integer. Let $X\subseteq E$ and $\langle X\rangle_{k-1} = E$. Then there is a $2.k$-flow ϕ on G with $S(\phi)\supseteq E-X$. If G is almost balanced then there exists such a k-flow.

Proof: We will prove, by induction on $|E-X|$, the existence of a flow ψ on G such that

*) $\quad \psi(e) \not\equiv 0 \mod k \qquad$ for all $e\in E-X$,

then lemma 2.1 will follow, using lemma 1.5.

If $E-X=\phi$ then the zero flow satisfies *). If $E \neq X$ then there is an elementary support S with $|S-X|\leq k-1$. Because $\langle X\cup S\rangle_{k-1} = \langle X\rangle_{k-1} = E$, by induction, there is a flow ψ' satisfying $\psi'(e)\not\equiv 0 \mod k$ for all $e\in E-(X\cup S)$. Take a principal flow f with support S. For an edge $e\in S-X$ consisting of half-edges h,h' we prove that there exists at most one integer p, $0\leq p\leq k-1$, such that

**) $\quad \psi'(h) + p.f(h) \equiv 0 \mod k.$

If p and p' satisfy **), then $(p-p').f(h) = k.(m-m')$ for suitable integers m and m'. Thus $|p-p'|.|f(h)|/k$ is an integer. Because $|f(h)|\in\{1,2\}$ and k is odd, $|p-p'|/k$ is an integer, too. It follows that if $0\leq p\leq p'\leq k-1$ then $p = p'$.

Because $|S-X|<k$, there exists p_0, $0\leq p_0\leq k-1$, satisfying **) for no edges of $S-X$. Then the flow $\psi = \psi' + p_0.f$ satisfies *).

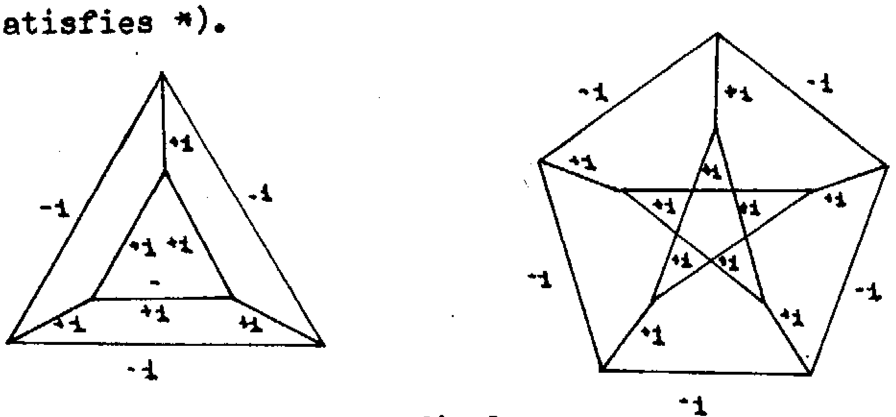

fig.1

III. Special subgraphs of bidirected graphs

3.1 Lemma: Let $G=(V,E,\sigma)$ be an unbalanced almost balanced bidirected graph without signed graphic isthmus. Then there are three unbalanced cycles C_1, C_2, C_3 such that $C_1 \cap C_2 \cap C_3 = \emptyset$ and every two of them have only a path in common. Moreover, $C_1 \cup C_2 \cup C_3$ is a support of a 3-flow.

Proof: Every elementary support in G is a balanced cycle. If e is in an unbalanced cycle C and C' is an elementary support containing e, then the set $C \cup C'$ contains two unbalanced cycles with only a path in common. One of them does not contain e, so for every e there is a balanced cycle C' and an unbalanced cycle C satisfying $e \in C'$ and $e \notin C$.

Let C_1, C_2 be two unbalanced cycles with only a path P in common and P is as small as possible. Let e be the first edge of P and $Y = C_1 \cup C_2 - \{x\}$. Y consist of a balanced cycle and of a path (maybe empty). Let C be an unbalanced cycle which does not contain e. We can consider the set C-Y is composed from paths P_i with only endpoints incident to Y. For every i we consider cycles cointained in $Y \cup P_i$. Suppose for a contradiction that all this cycles are balanced. We can put together C from these cycles by symmetric difference ($C \triangle C' = (C-C') \cup (C'-C)$). But symmetric difference of balanced cycles contains even number of edge-disjoint unbalanced cycles. This is a contradiction with the almost balancity of G.

Let $P_i \cup Y$ contain an unbalanced cycle. Because G is almost balanced the endpoints of P_i can not be incident to $C_i - C_j$ both at once, $\{i,j\} = \{1,2\}$. If both endpoints of P_i are incident to $P-\{x\}$, we obtain a contradiction with the minimality of P. Analogously we obtain a contradiction if one of enpoints is incident to $P-\{x\}$ and the second is incident to $(C_1 \cup C_2)-P$. Thus one endpoints is incident only to $C_1 - C_2$ and the second only to $C_2 - C_1$ and so P_i together with suitable parts of C_1 and C_2 is required cycle C_3.

Consider a graph G' consist only of the cycles C_1, C_2 and C_3. We can obtain from G', by switching and contraction of positive edges, a graph isomorfic to G'' on fig. 2a. On fig. 2b is a nowhere-zero 3-flow on this graph. As G' can be obtain from G'' by deleting and switching, by note 1.1 we can expend this flow to a 3-flow with support E(G').

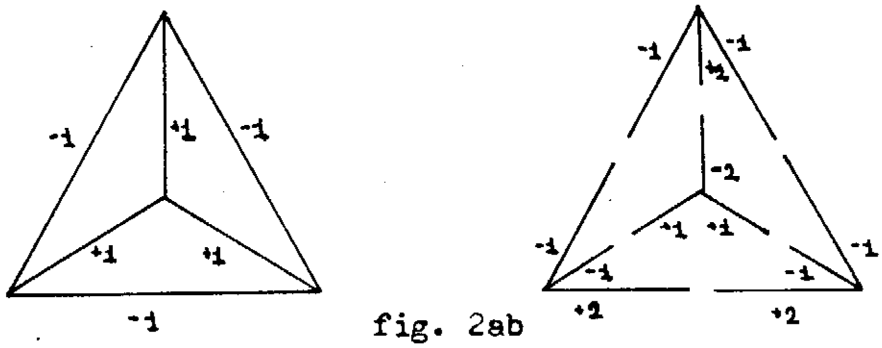

fig. 2ab

A 3-regular bidirected graph is called cellular tree if we can obtein it from a nonempty tree by following way: if v has degree k, k ≠ 3, we substitute it for an unbalanced cycle of length k such that the result has all vertices of degree 3. The same operation we do with any vertices of degree 3. On fig.3 there is this operation for k = 1,2,3,4.

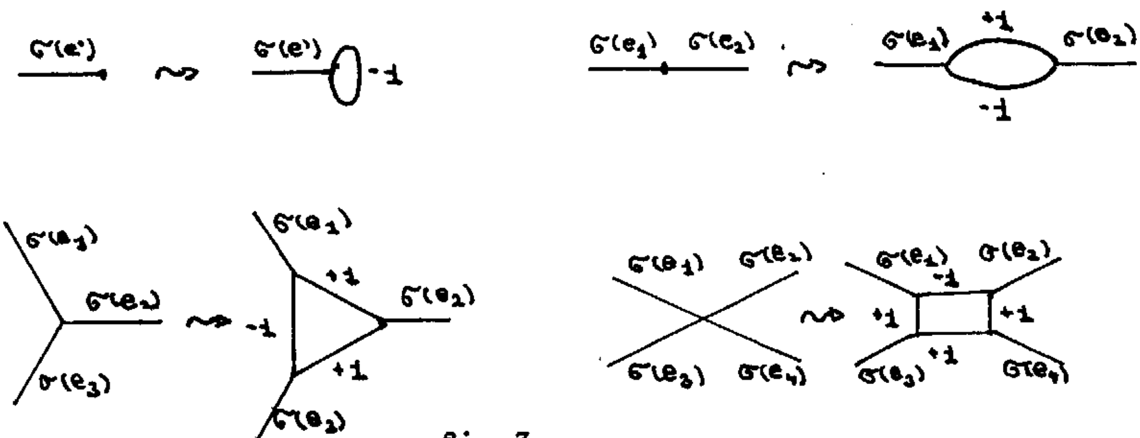

fig.3

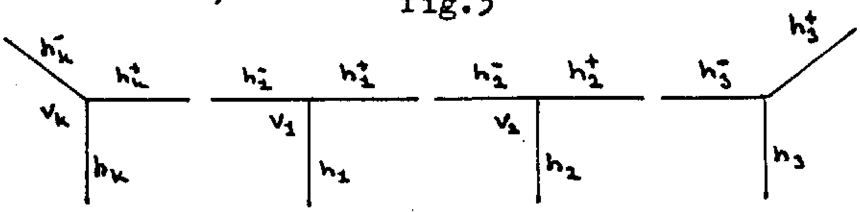

fig.4a

3.2 Lemma: There is a nowhere-zero 5-flow on every cellular tree $G=(V,E,\sigma)$.

Proof: We will construst a flow ϕ on G such that its values on unbalanced even cycles will be 1 or 3, on unbalanced odd cycles 1 or 2 and on the other edges 2 or 4. We start from a loop e' consisting of h',h'' incident to v' and we will proceed on the tree structure of G.

1. Put $\phi(h') = \phi(h'') = +1$, $\phi(h) = -2$ for h the third half-edge of $H(v')$.

2. If we constructed $\phi(h)$ and still did not construct $\phi(h')$ and $e_h = e_{h'}$, then put $\phi(h') = -\sigma(e_h).\phi(h)$.

3. If v is incident to distinct half-edges h,h',h'' such that no of them is in any unbalanced cycle and only $\phi(h)$ was constructed, then for $a\in\{+1,-1\}$
either $\phi(h) = 2.a$ and then put $\phi(h') = 2.a$, $\phi(h'') = -4.a$
or $\phi(h) = 4.a$ and then put $\phi(h') = -2.a$, $\phi(h'') = -2.a$.

4. Let v_1 be incident to distinct half-edges h_1, h_1^+, h_1^- and $e_{h_1^+}$, $e_{h_1^-}$ be in an unbalanced cycle $v_1,v_2,\dots,v_k,v_{k+1}= v_1$ and only $\phi(h_1)$ was constructed. Let the edge $e_i = \{v_i,v_{i+1}\}$ consists if half-edges h_i^+ and h_i^- and $H(v_i) = \{h_i, h_i^+, h_i^-\}$, fig.4a .

4a. Let k is odd and for $a\in\{+1,-1,+2,-2\}$ $\phi(h_1) = -2.a$. Put $\phi(h_1^+) = \phi(h_1^-) = a$ and for $i = 2,3,\dots,k$ put

$$*)\quad \begin{aligned} \phi(h_i^+) &= \phi(h_i^-) = a.\prod_{j=1}^{i-1} -\sigma(e_j) \\ \phi(h_i) &= -2.\ \phi(h_i^-). \end{aligned}$$

From *) follows $\phi(h_{i+1}^-) = a.\prod_{j=1}^{i} -\sigma(e_j) = a.(\prod_{j=1}^{i-1} -\sigma(e_j)).(-\sigma(e_i)) = \phi(h_i^+).-\sigma(e_i)$ for $i = 2,3,\dots,k-1$ and
$\phi(h_k^+) = a.\prod_{j=1}^{k-1} -\sigma(e_j) = a.(-1)^{k-1}.(\prod_{j=1}^{k}\sigma(e_j)).\sigma(e_k) = a.-\sigma(e_k) = -\sigma(e_k).\phi(h_1^-)$. Thus for edges and vertices of the unbalanced cycle the flow conditisinon are satisfied.

4b. Let k be even and $a \in \{+1,-1\}$.

If $\phi(h_1) = 2.a$ put (fig.4b)

$\phi(h_1^-) = a, \qquad \phi(h_1^+) = -3.a, \qquad \phi(h_2^-) = 3.a.\sigma(e_1),$

$\phi(h_2^+) = a.\sigma(e_1), \ \phi(h_2) = -4.a.\sigma(e_1).$

If $\phi(h_1) = -4.a$ put (fig.4c)

$\phi(h_1^-) = a, \qquad \phi(h_1^+) = 3.a, \qquad \phi(h_2^-) = -3.a.\sigma(e_1),$

$\phi(h_2^+) = a.\sigma(e_1), \ \phi(h_2) = 2.a.\sigma(e_1).$

For i = 3,4,...,k put

$$\phi(h_i^+) = \phi(h_i^-) = -a.\prod_{j=1}^{i-1} -\sigma(e_j)$$

$$\phi(h_1) = -2.\phi(h_i^-).$$

Because $\phi(h_{i+1}^-) = -a.\prod_{j=1}^{i} -\sigma(e_j) = a.\sigma(e_i).\prod_{j=1}^{i-1} -\sigma(e_j) =$
$= -\sigma(e_i).\phi(h_i^+)$ for i = 3,4,...,k-1 and because
$\phi(h_k^+) = -a.\prod_{j=1}^{k-1} -\sigma(e_j) = -a.\sigma(e_k).(-1)^{k-1}.\prod_{j=1}^{k}\sigma(e_j) = -a.\sigma(e_k) =$
$= -\sigma(e_k).\phi(h_1^-)$ the flow conditisions hold for all edges and vertices of the unbalanced cycle.

5. If v is incident to distinct half-edges h,h',h'' and $e_{h'} = e_{h''}$ is a loop and only $\phi(h)$ was constructed then put $\phi(h') = \phi(h'') = -\phi(h)/2$.

G is connected and so this construction assign a valuation for every half-edge.

A bidirected graph is general cellular tree if it can be obtain from a cellular tree by divisions, contractions of positive edges, switchings and by identifying some of its vertices (but not edges).

<u>3.3 Note:</u> Every general cellular tree is a support of a 5-flow.

This follows from note 1.1 and from the flow conditision for vertices.

3.4 Lemma: Let $G=(V,E,\sigma)$ be a connected graph with no vertices of degree one. Then there is a 2-edge-connected subgraph H such that at most one edge connects H to the rest of H in G.

Proof: Consider the bipartite graph T, $V(T) = I \cup B$, where I is the set of isthmuses of G and B is the set of components of G-I. Every member of B is a 2-edge-connected subgraph of G. For $i \in I$ and $b \in B$ $\{b, i\} \in E(T)$ iff i is incident to V(b). Every cycle in T induces a cycle in G which contain an isthmus of G. Thus T has no cycles. Every member of I has valency at least two in T because G has not vertices of degree one. Thus a subgraph $H \in B$ with valency at most one in T is connected at most one edge to the rest of H in G.

3.5 Lemma: Let H be an unbalanced connected subgraph of a bidirected graph G. Let P be a path in E(G)-E(H) with end vertices in V(H) and the other vertices not in V(H). Then there is an elementary support in $E(H) \cup P$ conteining P.

Sketch of proof: Every pair of vertices of an unbalanced graph are joined by edge progression with arbitrary sign (sign of edge progression $T = e_1, \dots, e_k$ is $\sigma(T) = \prod_{i=1}^{k} (e_i)$). Let T be edge progression joining end vertices of P such that $\sigma(T) = \sigma(P)$, the length of T is as small as possible and T has minimum number of various edges. Then edges of T together with P are required elementary support.

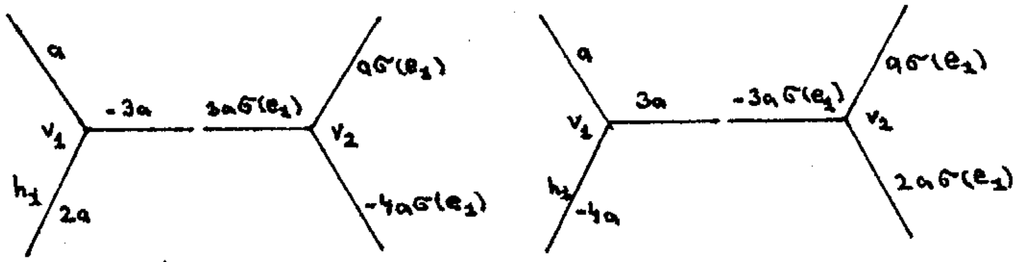

fig. 4bc

IV. Main result

<u>4.1 Lemma:</u> Let $G=(V,E,\sigma)$ be a connected unbalanced 3-unbalanced graph without signed graphic isthmus, without multiple edges with the same signs and without positive loops. Then there are edge-disjoint subgraphs $S_0, S_1, \ldots, S_k$ such that $Y = E(S_0) \cup E(S_1) \cup \ldots \cup E(S_k)$ is support of a 5-flow (if G is almost balanced then of a 3-flow) and $\langle Y \rangle_2 = E$.

Proof: If G is almost balanced we put S_0 the unbalanced graph from lemma 3.1. Is G is not almost balanced then it contains a general cellular tree. We put S_0 to be the general cellular tree which we can obtain from a cellular tree with maximum number of unbalanced cycle. In both these cases $E-E(S_0)$ contains no unbalanced cycle and no elementary support S, $|S| \leq 2$. Thus $\langle S_0 \rangle_2$ is connected.

We can choose maximum number of edge-disjoint supports of 2-flows $S_1, S_2, \ldots, S_k$ with $Y = \langle E(S_0) \cup E(S_1) \cup \ldots \cup E(S_k) \rangle_2$ connected. Suppose for a contradiction $Y \neq E$. Y is unbalanced and by lemma 3.5 every path with only end vertices incident to Y has length at least 3. Thus Y is edge-set of an induced subgraph and so $V(Y) \neq V$. Let H be a component of the graph induced by $V-V(Y)$, then H is balanced and has no vertices of degree one. Let K be the 2-edge-connected graph from lemma 3.4. G is 3-unbalanced and so there are at least tree edges joining K with the rest of G. Thus there are two edges e,e' joining K and V(Y). Let $v \neq v'$ are vertices of e and e' in V(K).

K is 2-edge-connected and there are two edge-disjoint path P,P' joining v and v'. These paths have the same sign and so $P \cup P'$ is support of a 2-flow. $P \cup \{e,e'\}$ is a part of elementary support in $Y \cup P \cup \{e,e'\}$ and so $\langle Y \cup P \cup P' \rangle_2$ is connected. It is a contradiction with the maximality of k.

Main theorem:
For every unbalanced bidirected graph without signed graphic isthmus there exists a nowhere-zero 30-flow.
For every unbalanced almost balanced graph without signed graphic isthmus there exists a nowhere-zero 9-flow.

Let us remark that Seymour [3] proved: for every balanced graph without signed graphic isthmus there exists a nowhere-zero 6-flow.

Proof: Let $k > 2$, suppose that G is an unbalanced graph without signed graphic isthmus with minimum number of E(G) which can not be provided with a nowhere-zero k-flow.
It is easy to see that G has no balanced loop and no multiple edges with the same signs. By lemma 1.4 G is 3-unbalanced.

By lemma 4.1 there exists $H \subseteq E(G)$ such that $\langle H \rangle_2 = E(G)$ and H is support of a 5-flow ϕ_1. By lemma 2.1 there is a 6-flow ϕ_2 with $S(\phi_2) \supseteq E(G)-H$. The flow $\phi = 6.\phi_1 + \phi_2$ has values on edges in absolute value smaller then
$6.\phi_1(e) + \phi_2(e) \leq 29$. So ϕ is a nowhere-zero 30-flow on G.

For unbalanced almost balanced graph we get by the same lemmas 3-flow ϕ_1 and 3-flow ϕ_2. Thus $\phi = 3.\phi_1 + \phi_2$ is a nowhere-zero 9-flow on G.

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