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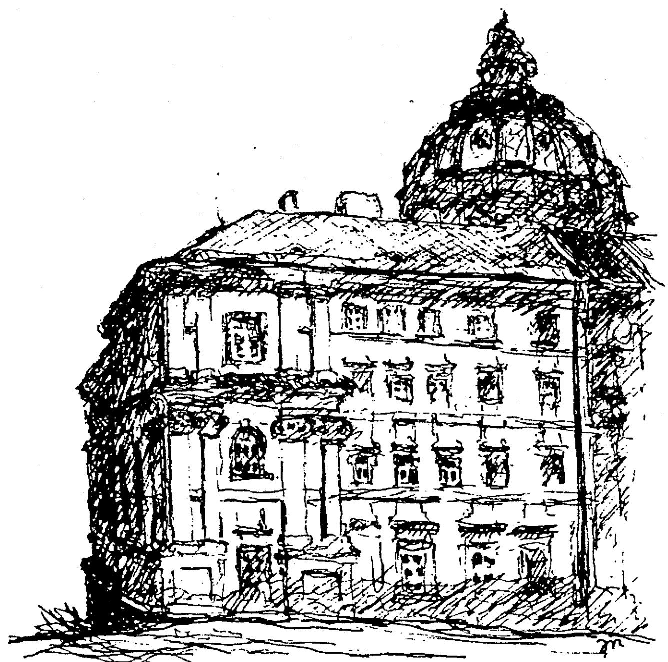
P. Bugata



Department of Applied Mathematics
Faculty of Mathematics and Physics
Charles University
(KAM MFF UK)
Malostranské nám. 25
118 00 Praha 1
Czechoslovakia

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ON ALGORITHMIC SOLVABILITY OF TRAHTENBROT-ZYKOV PROBLEM

Peter Bugata, Košice

I. Introduction

I.1 Trahtenbrot-Zykov problem.

We consider undirected graphs without loops and multiple edges. Let $V(G)$ be the vertex set and $E(G)$ the edge set of a graph G. For $v \in V(G)$ let $\Gamma(v)=\{u \in V(G): (u,v) \in E(G)\}$ and for $U \subseteq V(G)$ let G/U denote the subgraph of G induced by the set U. By the neighbourhood of a vertex v in G, denoted by $N(v,G)$, we mean the subgraph of G induced by the set of all vertices adjacent to v, i.e.

$$N(v,G)=G/\Gamma(v).$$

The neighbourhood set of G is the set

$$N(G)=\{N(v,G): v \in V(G)\},$$

while isomorphic graphs are considered as identical.

At the Smolenice symposium (1963) Zykov posed the following problem : For which finite graphs H does there exist a (finite) graph G with $N(G)=\{H\}$?

The origin of the interest to this problem lies in Trahtenbrot's investigations in automata theory yielding the name Trahtenbrot-Zykov problem (in the following T-Z problem). There are two modifications of T-Z problem, namely the finite (the graph G is required to be finite) and the infinite one.

T-Z problem has been studied by many authors, nevertheless its solution is known only for special classes of graphs such as paths, cycles, graphs homeomorphic to a star (for a survey see [2]). This fact led to the conjecture that T-Z problem is algorithmically unsolvable in the class of all graphs. The following theorem was proved in [1] by Bulitko.

Theorem A : There exists no algorithm which, given a finite graph H, will determine whether there exists a graph G with N(G)={H}.

An analogous result for the finite modification of T-Z problem is unknown till now. In the first part of the present paper we prove :

Theorem B : There exists no algorithm which, given a finite set N of finite graphs, will determine whether there exists a finite graph G whose neighbourhood set is N.

In the second part, the infinite modification of T-Z problem is studied and a new proof of Theorem A is given .

I.2 Post correspondence systems.

Bulitko proved Theorem A using the algorithmic unsolvability of "domino problem" (Kahr, Moore and Hao Wang, 1962). We use the classical "Post correspondence problem", which is also known to be algorithmically unsolvable (see for example [3]).

The Post correspondence system S is an ordered pair <A,B>, where A,B are n-tuples of non-empty finite words over the alphabet {a,b},

$$A=[u_1,\dots,u_n], \qquad B=[v_1,\dots,v_n].$$

A solution of S is a finite sequence of numbers $i_1,i_2,\dots,i_r$ from the set {1,...,n} such that

$$u_{i_1}u_{i_2}\dots u_{i_r}=v_{i_1}v_{i_2}\dots v_{i_r}.$$

The symbol uv denotes the concatenation of the words u and v.

Example 1: Let S=<A,B>, $A=[u_1,u_2,u_3]$ and $B=[v_1,v_2,v_3]$, where

u_1=ba $\qquad$ v_1=babb

u_2=bbaa $\qquad$ v_2=aaba

u_3=babbab $\qquad$ v_3=ab.

The system S has the solution 1,2,1,3, since $u_1u_2u_1u_3$= =babbaabababbab=$v_1v_2v_1v_3$.

The Post correspondence problem for S is to determine whether S has a solution. There is no algorithm which can solve Post problem in finitely many steps for any given Post system.

II. On the algorithmic solvability of the finite modification of T-Z problem.

First we will study the relationship between Post problem and the existence of labelled graphs with a prescribed neighbourhood set.

II.1 Labelled graphs and their neighbourhoods.

The labelled graph is an ordered pair (G,f), where G is a graph and f is a mapping of $V(G)$ on a finite alphabet Σ. If $v\in V(G)$, then $f(v)$ is called the colour of the vertex v. The f-neighbourhood of $v\in V(G)$ is defined as the rooted labelled graph

$$N_f(v,G)=(G/\Gamma(v)\cup\{v\},\ f/\Gamma(v)\cup\{v\})$$

with the root v. (f/U denotes the restriction of the mapping f to U.) By the f-neighbourhood set of a labelled graph (G,f) we understand the set

$$N_f(G)=\{N_f(v,G):\ v\in V(G)\}.$$

Two rooted labelled graphs are considered as identical, if there exists their isomorphism preserving the labelling of the vertices and mapping the root of one graph onto the root of the other.

Now we assign a finite labelled graph (G_S,f_S) to every Post system S having a solution and we describe a method how to assign to any Post system S a system of sets $\mathcal{M}(S)$ with the following properties:

1. If S has a solution, then $N_{f_S}(G_S) \in \mathcal{M}(S)$.

2. If there exists a finite labelled graph (G,f) such that $N_f(G)\in\mathcal{M}(S)$, then S has a solution.

II.2 Construction of the graph (G_S,f_S).

Let $S=\langle[u_1,\dots,u_n],[v_1,\dots,v_n]\rangle$ be a Post system with the solution $i_1,\dots,i_r$, $r\geq 2$. Let $w=x_1\dots x_r=y_1\dots y_r$, where $x_j=u_{i_j}$ and $y_j=v_{i_j}$ for $j=1,\dots,r$. We denote the characters of

w successively by $q_1,\dots,q_m$. The word w can be divided into disjoint parts corresponding to the words $x_1,\dots,x_r$ or $y_1,\dots,y_r$ and in this sense q_k is said to be a character (first character, last character) of the word x_j or y_j.

Now we describe the construction of a labelled graph (G_S,f_S), where $f_S:V(G_S) \to \Sigma$, $\Sigma=\{a,b,c_1,\dots,c_n,d_1,\dots,d_n, e,f,g,z\}$.

Let $V(G_S)$ contain the vertices $\bar{q}_1,\dots,\bar{q}_m,\bar{x}_1,\dots,\bar{x}_r, \bar{y}_1,\dots,\bar{y}_r$ coloured as follows :

$f_S(\bar{x}_j)=c_{i_j}$ and $f_S(\bar{y}_j)=d_{i_j}$, for every $j=1,\dots,r$;

if $q_k=a$, then $f_S(\bar{q}_k)=a$, otherwise $f_S(\bar{q}_k)=b$, for every $k=1,\dots,m$.

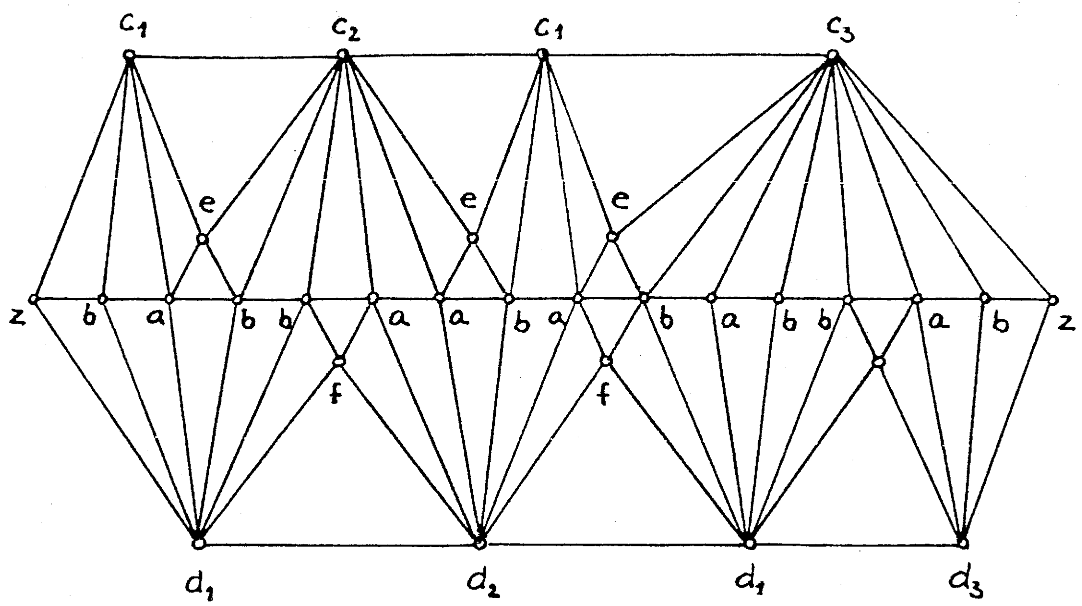
If q_k is a character of x_j or y_j, then $E(G_S)$ contains $(\bar{q}_k,\bar{x}_j)$ or $(\bar{q}_k,\bar{y}_j)$, respectively. Let the edges $(\bar{q}_k,\bar{q}_{k+1})$, $(\bar{x}_j,\bar{x}_{j+1})$, $(\bar{y}_j,\bar{y}_{j+1})$ for $k=1,\dots,m-1$, $j=1,\dots r-1$ belong to $E(G_S)$, too.

Furthermore, for every pair of the vertices $\bar{q}_k$, $\bar{q}_{k+1}$ such that q_k is the last character of x_j and q_{k+1} is the first character of x_{j+1}, we add an e-coloured vertex adjacent to $\bar{q}_k,\bar{q}_{k+1}$, $\bar{x}_j$ and $\bar{x}_{j+1}$. Similarly, for every k such that q_k is the last character of y_j and q_{k+1} is the first character of y_{j+1}, we add an f-coloured vertex adjacent to $\bar{q}_k$, $\bar{q}_{k+1}$, $\bar{y}_j$ and $\bar{y}_{j+1}$. Let the next vertices z_1,z_2 be z-coloured and let $(z_1,\bar{x}_1)$, $(z_1,\bar{y}_1)$, $(z_1,\bar{q}_1)$ and $(z_2,\bar{x}_r)$, $(z_2,\bar{y}_r)$, $(z_2,\bar{q}_m)$ belong to $E(G_S)$.

(For the Post system from Example 1 this part of G_S is sketched on Fig.1.)

Finally, we add g-coloured vertices $g_1,\dots,g_{r-1}$, edges $(z_1,g_1),(g_1,g_2),\dots,(g_{r-2},g_{r-1})$, (g_{r-1},z_2) and edges $(g_i,\bar{x}_i)$, $(g_i,\bar{x}_{i+1})$, $(g_i,\bar{y}_i)$, $(g_i,\bar{y}_{i+1})$ for $i=1,\dots,r-1$.

Fig. 1



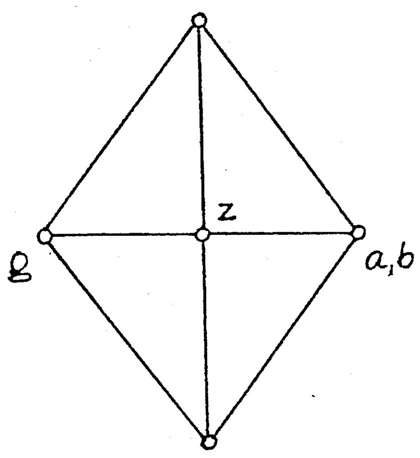
II.3 Definition of the system $\mathcal{M}(S)$.

By a wheel we mean a graph obtained from a cycle by adding one universal vertex, i.e. a vertex adjacent to all others vertices. Let $q,q_1,\dots,q_m\in\Sigma$ and $w=q_1q_2\dots q_m$. Symbol $W(q,w)$ denotes the rooted labelled graph isomorphic to the wheel with n+1 vertices, whose universal vertex is the q-coloured root and the next m vertices are labelled by $q_1,\dots,q_m$ successively.

Let $S=\langle[u_1,\dots,u_n],[v_1,\dots,v_n]\rangle$ be a Post system and let i,j,k be arbitrary values from the set $\{1,\dots,n\}$. Let $M_1(S)$ be the set of all graphs from Fig.2 and $M_2(S)$ the set of all graphs from Fig.3. At every vertex the set of its available colours is written . The "empty symbol" λ, written at a vertex v, means replacing v and incident edges by an edge, connecting vertices adjacent to v on the cycle.

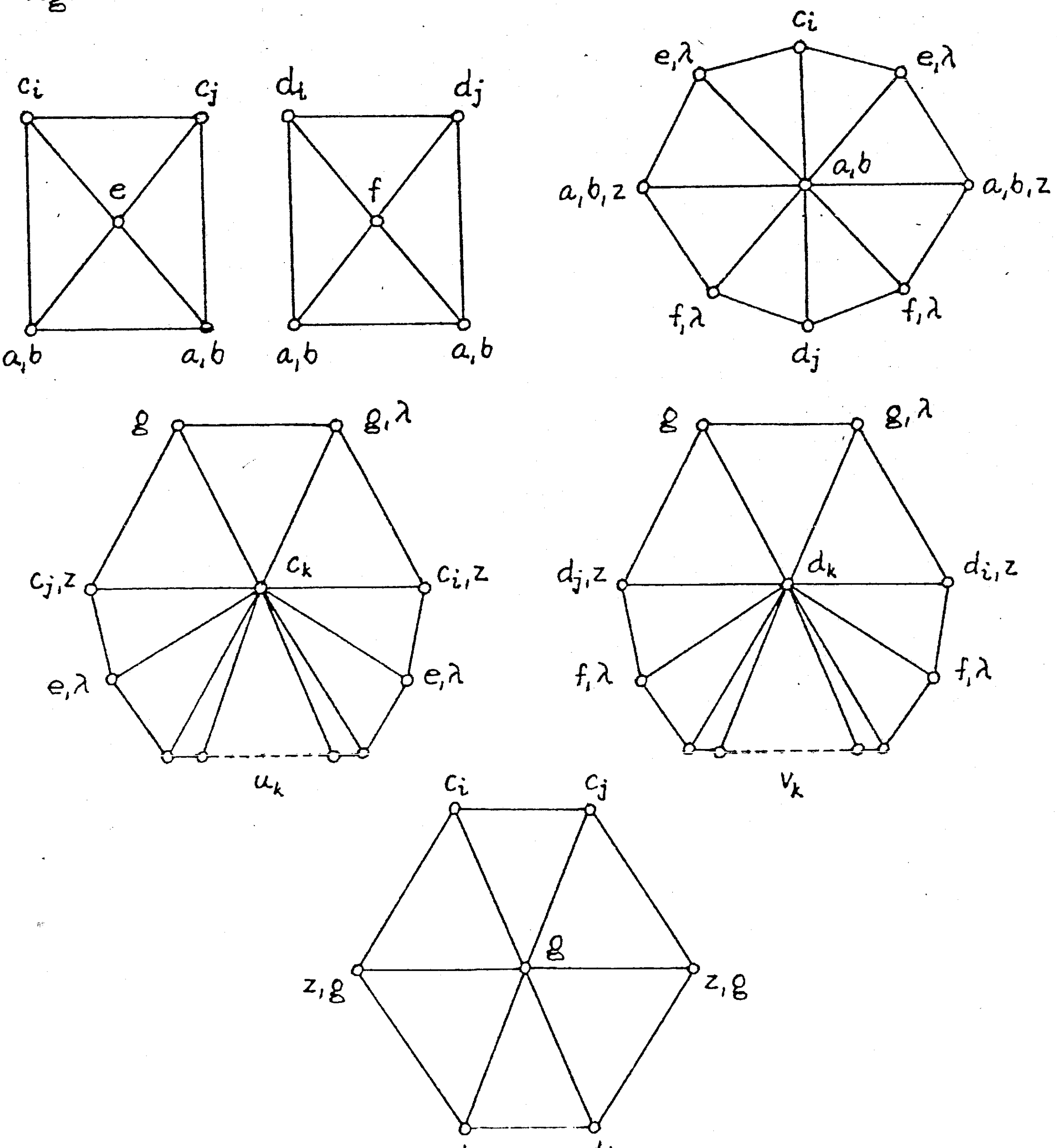
Next we give the exact description of $M_1(S)$ and $M_2(S)$.

Fig. 2



c_i
z
g
a,b
d_i

Fig. 3



c_i
c_j
e
a,b
a,b
d_i
d_j
f
a,b
a,b
c_i
e,λ
e,λ
a,b
a,b,z
a,b,z
f,λ
f,λ
d_j
g
g,λ
c_k
c_j,z
c_i,z
e,λ
e,λ
u_k
g
g,λ
d_k
d_j,z
d_i,z
f,λ
f,λ
v_k
c_i
c_j
g
z,g
z,g
d_i
d_j

Let $t_1,\dots,t_6$ substitute elements of Σ.

The set $M_1(S)$ contains all graphs $W(z,t_1c_igd_i)$, where $t_1\in\{a,b\}$.

The set $M_2(S)$ contains all following graphs :

$W(a,t_1t_2c_it_3t_4t_5d_jt_6)$, $W(b,t_1t_2c_it_3t_4t_5d_jt_6)$,

where $t_1,t_4\in\{a,b,z\}$, $t_2,t_3\in\{e,\lambda\}$, $t_5,t_6\in\{f,\lambda\}$;

$W(e,t_1c_jc_it_2)$, $W(f,t_1d_jd_it_2)$, where $t_1,t_2\in\{a,b\}$;

$W(g,t_1c_jc_it_2d_id_j)$, where $t_1,t_2\in\{z,g\}$;

$W(c_k,t_1t_2gt_3t_4u_kt_5)$,

where $t_1\in\{c_i,z\}$, $t_2\in\{g,\lambda\}$, $t_3\in\{c_j,z\}$, $t_4,t_5\in\{e,\lambda\}$;

$W(d_k,t_1t_2gt_3t_4v_kt_5)$,

where $t_1\in\{d_i,z\}$, $t_2\in\{g,\lambda\}$, $t_3\in\{d_j,z\}$, $t_4,t_5\in\{f,\lambda\}$.

Now we define $\mathcal{M}(S)$ as the system of all subsets of $M_1(S)\cup M_2(S)$, whose intersection with $M_1(S)$ is non-empty.

II.4 Neighbourhoods of labelled graphs and Post problem.

Lemma 1 : A Post system S has a solution if and only if there exists a finite labelled graph (G,f) such that $N_f(G)\in\mathcal{M}(S)$.

Proof : a) Let a system S have a solution. It is not difficult to see that the graph (G_S,f_S) has the desired properties.

b) Let $S=\langle[u_1,\dots,u_n],[v_1,\dots,v_n]\rangle$ and let there exist a finite labelled graph (G,f) such that $N_f(G)\in\mathcal{M}(S)$. Consider $G_o=G/\{v\in V(G):\ f(v)\in\{a,b,z\}\}$. The definition of the sets $M_1(S),M_2(S)$ implies that the degree of every a-coloured and b-coloured vertex in G_o is two and the degree of every z-coloured vertex is one. Let P be some component of G_o

be proved that $(y_j, y_{j+1}) \in E(G)$ for $j=1,\dots,m-1$. From the form of the available f-neighbourhoods of z_1 and z_2 it follows immediately that (z_1,x_1), (z_1,y_1), (z_2,x_r), $(z_2,y_m) \in E(G)$, too.

Claim 2 : a) $m=r$,

b) if $f(x_i)=c_j$, then $f(y_i)=d_j$ for $i=1,\dots,r$.

The cycle $z_1x_1x_2\dots x_rz_2y_my_{m-1}\dots y_1z_1$ is a component of the graph $G/\{v\in V(G):\ f(v)\in\{c_1,\dots,c_n,d_1,\dots,d_n,z\}\}$. Without loss of generality suppose that $r\le m$. Let g_1 be a g-coloured vertex adjacent to z_1. The definition of $M_2(S)$ implies that the f-neighbourhood of g_1 contains the vertices z_1,x_1,y_1,x_2,y_2 and the next vertex (denoted by g_2) adjacent to x_2 and y_2 and coloured by z or g. In the first case $g_2=z_2$ and thus $m=r=2$. In the second case we can repeat a similar consideration for g_2 as for g_1, etc. (See fig.4.) In this way we obtain the g-coloured vertices $g_1,\dots,g_{r-1}$ such that g_i is adjacent to x_i,x_{i+1},y_i,y_{i+1}.

Let g_r be adjacent to g_{r-1} and different from x_{r-1},x_r, y_{r-1}, y_r and g_{r-2} (or z_1). If $f(g_r)=g$, then the f-neighbourhood of g_r must contain a vertex adjacent to x_r, different from x_{r-1}. Since there is no vertex in G with this property, $f(g_r)=z$. As (g_r,x_r), $(g_r,y_r)\in E(G)$, necessarily $g_r=z_2$ and $m=r$. Moreover from the definition of $M_2(S)$ and from the form of the f-neighbourhoods of the vertices $g_1,\dots g_{r-1}$ we obtain that if $f(x_i)=c_j$, then $f(y_i)=d_j$ for $i=1,\dots r$.

Claim 2 and the definition of $M_2(S)$ imply that if $c_{i_1}c_{i_2}\dots c_{i_r}$ are colours of the vertices $x_1,x_2,\dots,x_r$, then containing a z-coloured vertex. (Its existence follows from $N_f(G) \cap M_1(S) \neq \emptyset$). Obviously, P is a path, whose endvertices, denoted for example by z_1 and z_2, are coloured by z and the next vertices by a or b.

Let $x \in V(G)$.

1. If $f(x) \in \{a,b,z\}$, then there exists the unique vertex adjacent to x labelled by a colour from $\{c_1, c_2, \dots c_n\}$.

2. If $f(x) \in \{c_1, c_2, \dots c_n\}$, then the restriction of the f-neighbourhood of x to the set of all vertices coloured by a,b,z is isomorphic to a path.

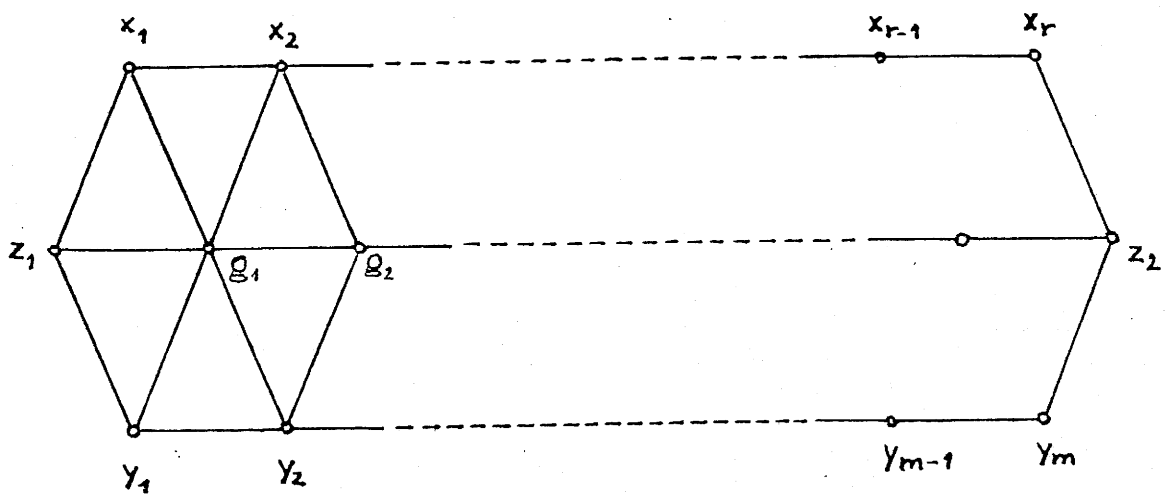
This implies that P can be divided into vertex disjoint paths $P_1, \dots, P_r$ so that there exist vertices $x_1, x_2, \dots, x_r$ in G with the following properties: $f(x_i) \in \{c_1, c_2, \dots c_n\}$ and $V(P_i) = \{v \in V(G): (v, x_i) \in E(G) \ \&\ f(v) \in \{a,b,z\}\}$. Similarly, P can be divided into vertex disjoint paths $P'_1, \dots, P'_m$ so that there exist vertices $y_1, y_2, \dots, y_m$ in G such that $f(y_i) \in \{d_1, d_2, \dots d_n\}$ and $V(P'_i) = \{v \in V(G): (v, y_i) \in E(G) \ \&\ f(v) \in \{a,b,z\}\}$. We observe still that $\bigcup_i V(P_i) = \bigcup_i V(P'_i) = V(P)$ and we number the parts $P_1, \dots, P_r$ and $P'_1, \dots, P'_m$ successively in the direction from z_1 to z_2.

Claim 1 : G contains the cycle $z_1 x_1 x_2 \dots x_r z_2 y_m y_{m-1} \dots \dots y_1 z_1$.

Let $q \in V(P_i)$ and $q' \in V(P_{i+1})$ be two adjacent vertices of P. The f-neighbourhood of q contains the vertices q' and x_i. As $(q', x_i) \notin E(G)$, the f-neighbourhood of q must contain an e-coloured vertex t and the edges (x_i, t), (t, q'). Then considering the available f-neighbourhoods of t, x_{i+1} is adjacent to t and $(x_i, x_{i+1}) \in E(G)$. In an analogous way, it can

$i_1, i_2, \ldots i_r$ is the solution of the Post system S and the colours of the vertices of P (except the z-coloured vertices) written successively create the word $u_{i_1} u_{i_2} \ldots u_{i_r} = v_{i_1} v_{i_2} \ldots v_{i_r}$.

Fig. 4



II.5 Labelled and unlabelled graphs with prescribed neighbourhoods.

In this part we show that the existence of labelled and unlabelled graphs with prescribed neighbourhoods of a special type is equivalent.

Before doing that we introduce some useful designations. We denote by $\deg_G(x)$ the degree of a vertex x in G, by card(M) the cardinality of a set M and by Z_1 the set of all integers greater than 1. Let (L, f_L) be a finite labelled graph (over the alphabet Σ) and let $\Phi : \Sigma \to Z_1$ be an one-to-one mapping . If $u \in V(L)$ is (is not) the root of L, then we denote by G_u a complete graph with $\phi(f_L(u))-1$ ($\Phi(f_L(u))$) vertices. Suppose that the vertex sets of graphs G_v for $v \in V(L)$ are disjoint.

Define a graph L_ϕ :

$V(L_\phi)= \bigcup_{v\in V(L)} V(G_v)$,

$E(L_\phi)= \bigcup_{v\in V(L)} E(G_v) \cup \{(x,y):\ x\in V(G_v) \& y\in V(G_u) \& (u,v)\in E(L)\}$.

If M is a set of rooted labelled graphs (over Σ), then $M_\phi=\{L_\phi:\ (L,f_L)\in M\}$.

Lemma 2 : Let $\phi : \Sigma \to Z_1$ be a one-to-one mapping.

If there exists a (finite) graph (G,f) such that $N_f(G)=M$, then there exists a (finite) graph G' such that $N(G')=M_\phi$.

Proof : Let G_v be a complete graph with $\phi(f_L(v))$ vertices for every $v\in V(G)$. (The sets $V(G_v)$ for $v\in V(G)$ are disjoint.) Furthermore, let

$V(G')= \bigcup_{v\in V(G)} V(G_v)$,

and $E(G')= \bigcup_{v\in V(G)} E(G_v) \cup \{(x,y):\ x\in V(G_v) \& y\in V(G_u) \& (u,v)\in E(G)\}$.

It is not difficult to see that $N(G')=M_\phi$.

The converse implication holds for the sets of the system $\mathcal{M}(S)$ and for the mapping ϕ with a particular property, too.

A one-to-one mapping $\phi : \Sigma \to Z_1$ is said to be good, if for every $q,q',q''\in\Sigma$:

$$\phi(q') + \phi(q'') > \phi(q).$$

Lemma 3 : Let S be a Post system, $M\in\mathcal{M}(S)$ and $\phi : \Sigma \to Z_1$ be good.

If there exists a (finite) graph G' such that $N(G')=M_\phi$, then there exists a (finite) labelled graph (G,f) such that

$N_f(G)=M$.

Proof : Let $N(G')=M$. Define a binary relation "$\equiv$". For every $x,y \in V(G')$:

$$x\equiv y \Leftrightarrow x=y \vee [(x,y)\in E(G') \ \& \ \Gamma(x)-\{y\}=\Gamma(y)-\{x\}]$$

It is easy to see that "$\equiv$" is the equivalence relation. Let $V(G')/\equiv$ be the partition of $V(G)$ into equivalence classes and for $x\in V(G')$ $[x]$ be the equivalence class containing x. The factor graph G is defined as follows :

$$V(G)=V(G')/\equiv ,$$

$$([x],[y])\in E(G) \Leftrightarrow (x,y)\in E(G').$$

The validity of this definition follows from the properties of "$\equiv$".

Consider the graph $N(x,G')$, where x is any vertex of G'. $N(x,G')$ is isomorphic to a graph $L_\phi \in M_\phi$. The vertex set of $N(x,G')$ can be decomposed to disjoint sets U_v, $v\in V(L)$, corresponding to vertex sets of the graphs G_v, $v\in V(L)$. (See the definition of L_ϕ.)

Let w be an universal vertex of L, $y\in U_v$ for $v\in V(L)$ and $z\equiv y$. The definition of "$\equiv$" implies that if $z\neq x$, then $z\in V(N(x,G'))$. Moreover as L is a wheel, whose number of vertices is greater than 4, either $z=x$ and $v=w$ or $z\in U_v$. Thus we obtain that the sets U_v, $v\in V(L)-\{w\}$ and $U_w\cup\{x\}$ are unions of some equivalence classes.

Now we prove that $U_w\cup\{x\}=[x]$. Obviously, $[x]\subseteq U_w\cup\{x\}$. Suppose that there exists of $y\in U_w\cup\{x\}-[x]$ and thus $V(N(x,G'))$ is a proper subset of $V(N(y,G'))$. $N(y,G')$ is isomorphic to some $L'_\phi \in M_\phi$ and $V(N(y,G'))$ can be decomposed to disjoint subsets W_v, $v\in V(L')$, corresponding to vertex sets of G_v, $v\in V(L')$. Vertices belonging to different sets of the partition into U_v, $v \in V(L)$, belong also to different sets of the partition into W_v, $v \in V(L')$. This implies that if $[x] \subseteq W_u$, then u is not an universal vertex of L' and $deg_{L'}(u) \geq deg_L(\mathbf{w}) \geq 4$ - a contradiction : L' is a wheel.

For every $x \in V(G')$ we obtain :

$$card([x]) = card(U_w \cup \{x\}) = (\phi(q)-1) + 1 = \phi(q)$$

for some $q \in \Sigma$. Now we define the labelling f of G :

$$f([x]) = \phi^{-1}(card([x])) \text{ for } x \in V(G').$$

Finally, we show that if x is an arbitrary vertex of G' and $N(x,G') \cong L_\phi$, then $N_f([x],G)$ is isomorphic to (L, f_L). Let U_v, $v \in V(L)$, be defined than before and w is the root of L. Obviously, it is sufficient to prove that the sets U_v are elements of $V(G')/\equiv$. Suppose that for some $v \in V(L)-\{w\}$ U_v contains two different equivalence classes [y] and [z]. Then

$$card(U_v) \geq card([y]) + card([z]).$$

According to the previous consideration $card([y]) = \phi(q')$, $card([z]) = \phi(q'')$ and according to the definition of L_ϕ $card(U_v) = \phi(q)$ $(q, q', q'' \in \Sigma)$, which is a contradiction to ϕ is good.

As x is an arbitrary vertex of G', $N_f(G)=M$.

Corollary 1 : Let S be a Post system and $\phi : \Sigma \to Z_1$ be good.

Then S has a solution if and only if there exists a finite graph G such that $N(G) \in \{M_\phi : M \in \mathcal{M}(S)\}$.

From the definition of the system $\mathcal{M}(S)$, and the sets M_ϕ and from the algorithmic unsolvability of the Post problem we obtain :

Theorem 1 : There exists no algorithm which, given a finite set N of finite graphs, will determine whether there exists a finite graph G with N(G)=N.

III. On algorithmic solvability of the infinite modification of T-Z problem.

In a similar way as in part II for the finite modification of T-Z problem we will study the relationship between the existence of infinite graphs with prescribed neighbourhoods and the "infinite" modification of Post problem and prove the known Bulitko´s result.

III.1 The "infinite" modification of the Post problem.

Let $S=\langle[u_1,\dots,u_n],[v_1,\dots,v_n]\rangle$ be a Post system. An ω-solution of S is an infinite sequence of numbers $i_1,i_2,i_3,\dots$ from the set $\{1,\dots,n\}$ such that

$$\prod_k u_{i_k} = \prod_k v_{i_k} ,$$

where by $\prod_k u_{i_k}$, $\prod_k v_{i_k}$ we mean the infinite concatenation of these words. Obviously, if a Post system S has a solution, then S has an ω-solution, too.

The "infinite" Post correspondence problem for S is to determine whether S has an ω-solution. Using a modification of the proof of the algorithmic unsolvability of the Post problem can be proved that there is no algorithm which can solve the "infinite" Post correspondence problem for any given Post system.

Lemma 4 : A Post system S has an ω-solution if and only if there exists labelled graph (G,f) such that $N_f(G)\in\mathcal{M}(S)$.

Proof : a) If a Post system S has an ω-solution, then in a similar way as in II.2 we can assign to it the infinite labelled graph (G_s,f_s) such that $N_{f_s}(G_s)\in\mathcal{M}(S)$. ($G_s$ contains only one z-coloured vertex.)

b/ Let (G,f) be a labelled graph with $N_f(G)\in\mathcal{M}(S)$ and F be some component of $G_o=G/\{v\in V(G):\ f(v)\in\{a,b,z\}\}$ containing a z-coloured vertex. If F is finite, then S has a solution and thus also an ω-solution. Otherwise, similarly as in the proof of Lemma 1, we can prove that F is a one-way infinite path and find the infinite sequence $i_1,i_2,i_3,\ldots$ such that the colours of the vertices of F (except the endvertex) written successively create the word $\prod_k u_{i_k} = \prod_k v_{i_k}$.

As Lemma 2 and Lemma 3 hold also for infinite graphs, we obtain :

Corollary 2 : Let S be a Post system and $\phi\ :\Sigma\ \rightarrow\ Z_1$ be good.

S has an ω-solution if and only if there exists a graph G such that $N(G)\in\{M_\phi,\ M\in\mathcal{M}(S)\}$.

Corollary 3 : There exists no algorithm which, given a finite set N of finite graphs, will determine whether there exists a graph G with N(G)=N.

III.2 The infinite modification of T-Z problem and graphs with prescribed neighbourhood set.

We consider only finite neighbourhood sets containing finite graphs. It is sufficient to confine to graphs with countable vertex sets, since for every infinite graph there exists a countable graph with the same neighbourhood set.

Let $\bigcup_{i=1}^{n} G_i$ denote the disjoint union of graphs $G_1,\dots,G_n$.

Lemma 5 : If there exists a (finite) graph G' such that $N(G')=\{\bigcup_{i=1}^{n} H_i\}$, then there exists a (finite) graph G with $N(G)=\{H_1,\dots,H_n\}$.

Proof : Let G' satisfy the assumption of the lemma, and for every $x\in V(G')$ $N(x,G')=\bigcup_{i=1}^{n} H_i(x)$, where $H_i(x)\cong H_i$, $i=1,\dots,n$. If there exist more numberings of $H_1(x),\dots,H_n(x)$ with this property, take one of them. We define a graph G as follows :

$V(G) = \{[x,i]: x\in V(G')\ \&\ i\in\{1,\dots n\}\};$

$([x,i],[y,k])\in E(G) \iff x\in V(H_k(y))\ \&\ y\in V(H_i(x)).$

Let [x,i] be an arbitrary vertex of G. We show that the mapping $f : V(N([x,i],G)) \to V(H_i(x))$ such that

$f([y,k]) = y$

is an isomorphism $N([x,i],G)$ to $H_i(x)$.

For every $y\in V(H_i(x))$ there exists the unique $k\in\{1,\dots,n\}$ such that $x\in V(H_k(y))$, and thus $[y,k]\in V(N([x,i],G))$. This implies that f is a bijection.

If $([y,k],[z,l])$ is an edge of $N([x,i],G)$, then $(y,z)\in E(G')$. As $H_i(x)$ is an induced subgraph of G', the edge (y,z) belongs to $E(H_i(x))$, too. Conversely, let $(y,z)\in$ $\in E(H_i(x))$ and [y,k], [z,l] be vertices of $N([x,i],G)$. Then

$x \in V(H_k(y))$ and $(y,z),(x,z) \in E(G')$ implies $z \in V(H_k(y))$. The fact that $y \in V(H_l(z))$ can be proved alike. We obtain $([y,k],[z,l]) \in E(G)$.

Thus f is the isomorphism $N([x,i],G)$ to $H_i(x)$ and the arbitrariness in the choice of $[x,i]$ implies $N(G)=\{H_1,\dots,H_n\}$.

For infinite graphs the converse implication holds, too.

Lemma 6 : If there exists a graph G with $N(G)=\{H_1,\dots,H_n\}$, then there exists a graph G' such that $N(G')=\{\bigcup_{i=1}^{n} H_i\}$.

Proof : Let $N(G)=\{H_1,\dots,H_n\}$. Suppose that the set $\{x \in V(G) : N(x,G) \cong H_i\}$ is infinite, otherwise we can consider infinite number of disjoint copies of G. If $n>2$, then we denote by τ and ξ two bijections Z (Z is the set of all integers) to V(G) such that :

if $i \equiv 1 \bmod 3$, then $N(\tau(i),G) \equiv H_1$ and $N(\xi(i),G) \equiv H_2$;

if $i \equiv 2 \bmod 3$, then $N(\tau(i),G) \equiv H_2$ and $N(\xi(i),G) \equiv H_1$;

if $i \equiv \emptyset \bmod 3$, then neither $N(\tau(i),G)$ nor $N(\xi(i),G)$ are isomorphic to H_1 and H_2.

Now we define a graph G'. Let $V(G')=Z \times Z - \{[i,j] : i \equiv \emptyset \bmod 3$ & $j \equiv \emptyset \bmod 3\}$ and let $E(G')$ contain the edges :

$([i,j],[i,k])$ for $i \equiv 1 \bmod 3$ & $(\tau(j),\tau(k)) \in E(G)$ and for $i \equiv 2 \bmod 3$ & $(\xi(j),\xi(k)) \in E(G)$;

$([j,i],[k,i])$ for $i \equiv 1 \bmod 3$ & $(\xi(j),\xi(k)) \in E(G)$ and for $i \equiv 2 \bmod 3$ & $(\tau(j),\tau(k)) \in E(G)$.

If i,j are not divisible by 3, then $N([i,j],G') \cong H_1 \cup H_2$. The neighbourhoods of the others vertices of G' are isomorphic to H_j for some $j \in \{3,\dots,n\}$, and thus $N(G') = \{H_1 \cup H_2, H_3, \dots, H_n\}$. By a repeat of this procedure we obtain a graph G" with $N(G'') = \{\bigcup_{i=1}^{n-1} H_i, H_n\}$. So it is sufficient to prove the lemma for n=2.

Now let $N(G) = \{H_1, H_2\}$ and τ, ξ be two bijections Z to V(G) with the following properties:

if $i \equiv 1 \bmod 2$, then $N(\tau(i),G) \equiv H_1$ and $N(\xi(i),G) \equiv H_2$;

if $i \equiv \emptyset \bmod 2$, then $N(\tau(i),G) \equiv H_2$ and $N(\xi(i),G) \equiv H_1$.

Let $V(G') = Z \times Z$ and let E(G') contain the edges:

$([i,j],[i,k])$ for $i \equiv 1 \bmod 2$ & $(\tau(j),\tau(k)) \in E(G)$ and for $i \equiv \emptyset \bmod 2$ & $(\xi(j),\xi(k)) \in E(G)$;

$([j,i],[k,i])$ for $i \equiv 1 \bmod 2$ & $(\xi(j),\xi(k)) \in E(G)$ and for $i \equiv \emptyset \bmod 2$ & $(\tau(j),\tau(k)) \in E(G)$.

It is not difficult to see that N(x,G') is isomorphic to $H_1 \cup H_2$ for every $x \in V(G')$.

Corollary 4: A graph G' with $N(G') = \{\bigcup_{i=1}^{n} H_i\}$ exists if and only if there exists a graph G with $N(G) = \{H_1, \dots, H_n\}$.

Remark: Lemma 6 may not be true for finite graphs. For example if $G = K_2 + (K_1 \cup K_1)$, then $N(G) = \{K_2, P_3\}$, but there is no finite graph G' with $N(G') = \{K_2 \cup P_3\}$. (K_n, P_n is a complete graph, path, respectively (with n vertices), and "+" denotes Zykov sum of graphs.)

According to Corollary 3, Lemma 5 and Lemma 6 we obtain:

Theorem 2 : There exists no algorithm which, given a finite graph H, will determine whether there exists a graph G with $N(G) = \{H\}$.

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