HOMO
NOLO
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Šámalova chata, Nová Louka

November 6 – 10

Eva Ondráčková (ed.)

Contents

Preface

The Workshop on Graph **Homo**morphisms and Geometric Methods at **No**vá **Lo**uka has gradually evolved into the autumn fixed point of KAM/DIMATIA/ITI academic life, to complement the traditional Spring Schools and Midsummer Workshops. As the youngest of this Prague Combinatorial Trinity it has waited for a mini-proceedings in our Preprint Series for a while, but with its 5th continuation in 2006 it is time to look back and compare the previous issues of the workshop.

The first Homonolo was (probably) organized in 2000, the International Year of Mathematics (I say probably because its web page has not survived and the photographs from pre-digital times do not carry dates). Memories are vague, except for a few glimpses of Jarik Nešetřil giving his talk on a flip chart hung on huge antlers (Nová Louka is a hunters' lodge).

The December 2002 workshop with distinguished invited speakers was the first official event of the DDR project (no connection to Eastern Germany despite of the geographical location, the abbreviation stands for the DIMATIA-DIMACS-Rényi NSF grant).

The year 2003 was so far the only one when we could enjoy the Indian summer – Homonolo III was organized in September. Apart from research talks we made full use of the weather and enjoyed daily swimming in the nearby reservoir Černá Nisa. The conference photo in swim suites documents well the warm climate.

The man in charge of Homonolo 2005 was professor Jiří Matoušek, and Homonolo IV turned into a problem solving olympiad. The most successful team was led by Anna de Mier and their solution of one enumeration problem was then presented at the 6th Czech-Slovak Symposium on Combinatorics in July 2006 in Prague.

Last Homonolo's atmosphere adhered more to the original scenario of an ongoing problem session – presentations of open problems with solid background information. As a result of tedious work of the editor Eva Ondráčková you can now see most of the problems and presentations.

The atmosphere of Homonolos is unique. It is definitely affected by the surrounding nature, magnificent woods and marshes of Jizera Mountains. Nowhere else would afternoon hikes in fog and hail storms be tempting, but here they simply belong to the warmly accepted daily schedule. And occasional hikes in several inches of fresh snow are just a cherry on the cake. But the inside chambers of the historic lodge (formerly adjoint to a small glass factory) with granite door frames and two hundred years old wooden rafters create a warming-up atmosphere if the weather outside is too chilly. And the fantastic skills of the chef Mr. Polák make every day a feast. The warmth and friendliness of the staff is one of the reasons why Homonolo will be for a long time stationed in one place.

Time seems frozen here when we are coming back year after year to the well known and friendly place. Yet there is one thing that reminds us of the inevitable flow of time. During the seven years we have met three generations of St. Bernard dogs guarding the lodge. Old Cesar, young Clea I and a puppy Clea II. Let us dedicate these mini-proceedings to those barking local co-organizers of our workshops.

Jan Kratochvíl

Cesar

Clea I

Clea II

Locally constrained homomorphism of planar graphs

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We consider finite undirected graphs without loops or multiple edges. A graph homomorphism between graphs G and H is an edge-preserving vertex mapping from G to H, formally, $f : V(G) \longrightarrow V(H)$ is a homomorphism if $f(u)f(v) \in E(H)$ whenever $uv \in E(G)$. Graph homomorphisms are often referred to as generalized graph colorings, since proper k-colorings of G are exactly homomorphisms from G into the complete graph K_k.

It follows from the definition that if $f : V(G) \longrightarrow V(H)$ is a homomorphism and $u \in V(G)$ an arbitrary vertex of G, the neighborhood of u is mapped into the neighborhood of $f(u)$, formally,

$$f(N_G(u)) \subseteq N_H(f(u)).$$

The notion of locally constrained homomorphisms is based on this observation. The homomorphism f is called *locally bijective* (*injective*) if for every $u \in V(G)$, the restricted mapping $f_u : N_G(u) \longrightarrow N_H(f(u))$ is bijective (injective, respectively).

Locally bijective homomorphisms have been studied for many years under the notion of *graph covers* from the point of view of topological and algebraic graph theory. They also play an important role in the theory of local computations.

Locally injective homomorphisms, also known as *partial covers* are closely related to distance constrained graph labelings (stemming from the Frequency Assignment Problem). An $L(2,1)$-labeling of a graph G is a mapping $\lambda : V(G) \longrightarrow \{0, 1, \ldots, k\}$ such that

- $|\lambda(u) - \lambda(v)| \geq 2$ whenever $uv \in E(G)$, and
- $\lambda(u) \neq \lambda(v)$ whenever u and v have a common neighbor in G.

The k is called the span of the labeling, and $L_{2,1}(G)$ is the minimum possible span of an $L(2,1)$-labeling of G. It is easy to show that $L_{2,1}(G) \leq k$ if and only if G allows a locally injective homomorphism into the complement of the path of length k.

The study of the computational complexity of these questions was initiated in [1, 5]. From this point of view, we are interested in these decision problems

H-LBHom	H-LIHom
Input: A graph G.	Input: A graph G.
Question: Does G allow a locally bijective homomorphism into H?	Question: Does G allow a locally injective homomorphism into H?

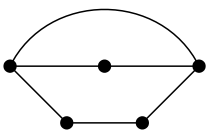
Note that in both cases, H is a fixed parameter of the problem. The goal is to give a full characterization for which graphs H the problems are polynomial time solvable and for which they are NP-complete (P $\neq$ NP assumed). In particular, it is an open problem whether the polynomial/NP-complete dichotomy holds for these problems. The known complexity results for H-LBHom can be found in [1, 10, 11, 12] and for H-LIHom in [5, 6, 8] and implicitly (in the terms of $L(2,1)$-labeling) in [4, 2]. It is known that H-LBHom is NP-complete for regular graphs H of valency at least three, and polynomial time solvable for graphs with at most 2 vertices in each block of the degree partition[1] and for the so called Theta graphs (cf. the next paragraph). It is also known [6] that for every graph H, H-LIHom is at least as difficult as H-LBHom, and hence Theta graphs are the first candidates for systematic research of the complexity of H-LIHom. It is rather interesting that already among such simple looking graphs, nontrivial polynomial time solvable cases as well as NP-complete ones are found, and the full characterization is not known. A small breakthrough in this area is the recent result [7] which gives a full characterization of the computational complexity of the list version of H-LIHom. It is also noted there that all NP-complete cases remain NP-complete when the input graph is planar. Besides of this result, very little is known about the complexity of locally constrained homomorphisms for planar graphs (the only other result is from [2] stating that $L_{2,1}$-labeling of span 8 is NP-complete for planar graphs). Thus we formally propose

Problem 1. *Study the computational complexity of H-*LBHom *and H-*LIHom *for planar inputs.*

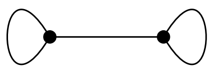
[1]The degree partition of H is the coarsest partition of its vertex set $V(H) = \bigcup_{i=1}^{k} V_i$ such that every vertex in V_i has exactly m_{ij} neighbors in V_j, for every i, j, for some matrix $M = (m_{ij})_{i,j=1}^{k}$ called the degree refinement matrix of H. The degree partition is unique (with a canonical lexicographic ordering of the sets of the partition) and can be found in polynomial time. A necessary condition for the existence of a locally bijective homomorphism from G to H is that G and H have the same degree refinement matrix.

It would be a daring conjecture, but not totally improbable, that for every graph H, the complexities of H-LIHom for planar inputs and for general inputs are the same (the positive answer would imply that $L(2,1)$-labeling of span 4 is NP-complete for planar graphs, a long standing open problem). However, for the locally bijective homomorphisms, such result would be even more surprising.

It is time now to define two types of graphs explicitly. The *Theta* graphs are graphs $\Theta(n_1, n_2, \ldots, n_k)$ which consist of two vertices of degree k, with the remaining vertices having all degree 2 and being placed on paths joining the two master vertices (the lengths of the paths are $n_1, n_2, \ldots, n_k$). To keep the graphs simple (no multiple edges), we assume that 1 occurs at most once among $n_1, n_2, \ldots, n_k$. Observe that $\Theta(1, 2, 3)$ is the complement of the path of length 4, and hence the graphs having the $L_{2,1}$-span at most 4 are exactly the graphs allowing a locally injective homomorphism into $\Theta(1, 2, 3)$.



$\Theta(1, 2, 3)$



$W(1; 1; 1)$

The *Weight* graphs are graphs $W(a_1, \ldots, a_k; b_1, \ldots, b_l; c_1, \ldots, c_m)$ consisting of two master vertices, one of degree $2k + l$ and the other one of degree $l + 2m$, with the remaining vertices having all degree 2 and being placed on cycles of lengths $a_1, \ldots, a_k$ containing the first master vertex, on cycles of lengths $c_1, \ldots, c_m$ containing the second master vertex, and on paths of lengths $b_1, \ldots, b_l$ connecting the two master vertices. To avoid multiple edges and loops we should require that there is at most one 1 among $b_1, \ldots, b_l$, and that there are no 1's among $a_1, \ldots, a_k, c_1, \ldots, c_m$. However, with apologies to the reader, we make an exception here. We will consider the weight graph $W(1; 1; 1)$ (which happens to be the complete graph K_2 with one loop at each of its vertices) and we describe explicitly what locally constrained homomorphisms to this graph mean. These in fact can be quite conveniently described as special colorings. A coloring (not necessarily proper) of the vertices of G with 2 colors is a locally bijective homomorphism onto $W(1; 1; 1)$ if and only if every vertex has exactly two neighbors of its own color and exactly one neighbor of the other color (a necessary condition is that G is cubic). A coloring of the vertices of G with 2 colors is a locally injective homomorphism into $W(1;1;1)$ if and only if every vertex has at most two neighbors of its own color and at most one neighbor of the other color (a necessary condition is that G is subcubic).

The complete, Theta and weight graphs play an important role in reductions among locally constrained graph homomorphism problems. Therefore we propose to start the work on Problem 1 with these graphs.

Problem 2. *Fill in the computational complexities in this table*

H-	Planar-H-LIHom	Planar-H-LBHom
K_4	*NP-complete*	
$W(1;1;1)$		
$\Theta(1,2,3)$		*Poly*

Comments

The complexity of K_4-LIHom for planar inputs was asked as an open problem at Homonolo'06, and was proved NP-complete by M. Siggers. However, it turned out to have been known [3] in the connection to coloring distance powers of graphs. Indeed, the second distance power of a graph G is k-colorable if and only if G allows a locally injective homomorphism into K_k. The NP-completeness result has later been then extended to arbitrary $k \geq 4$ in [13].

Locally bijective homomorphisms onto K_k are related to perfect codes in graphs. An independent set $C \subset V(G)$ of vertices is a perfect code in G if every vertex in $V(G) \setminus C$ has exactly one neighbor in C. It is then easy to see that a $(k-1)$-regular graph admits a locally bijective homomorphism onto K_k if and only its vertex set can be partitioned into k perfect codes (the preimages of the vertices of K_k in the homomorphism). The reduction showing the NP-completeness of K_4-LBHom presented in [9] goes from edge-3-colorability, whose complexity for planar cubic graphs is open (every bridgeless planar cubic graph is edge-3-colorable as a consequence of the Four Color Theorem, but it is a well known yet open conjecture that a planar bridgeless subcubic graph is not edge-3-colorable if and only if it arises from a cubic graph by subdivision of exactly one edge by a vertex of degree 2). On the other hand, it is proved in [9] that the existence of a perfect code is NP-complete for planar cubic graphs, but the reduction does not seem to extend to partitions into perfect codes.

Given a cubic graph G, transform it into a subcubic G' by replacing every edge by a path of length three. Then G admits a locally bijective homomorphism onto $W(1;1;1)$ if and only if G' admits a locally injective homomorphism into $\Theta(1,2,3)$ [4]. The transformation preserves planarity, and hence NP-completeness of $W(1;1;1)$-LBHom (for planar graphs) would imply NP-completeness of $\Theta(1,2,3)$-LIHom (for planar graphs). And of course NP-completeness of $W(1;1;1)$-LIHom as well.

On the other hand, an algorithm for planar inputs could take advantage of the existence of dual graphs. The dual of a cubic planar graph is a planar triangulation. Assume a bicoloring of the vertices of the primary graph such that every vertex has exactly one neighbor of the other color. Such bicolored edges form a cut, and the corresponding dual edges form an even subgraph in the dual graph. Thus the following is the dual reformulation of Planar-$W(1;1;1)$-LBHom.

Problem 3. *What is the complexity of deciding whether a planar triangulation contains a spanning even subgraph which meets every face of the triangulation in exactly one edge?*

Had the request 'exactly one edge' been replaced by 'odd number of edges', a system of equations modulo 2 could be easily used to solve the problem in polynomial time.

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Triangulation of the unit square into an odd number of triangles

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Günter Ziegler asked the following question: Let δ be the difference of the area of the largest and the smallest triangle in a triangulation of the unit square into n triangles where n is odd. How small can δ be (depending on n of course)?

My question: Let $t_i(\varepsilon) = \log\left(\frac{1-(i-1)\varepsilon}{1-i\varepsilon}\right)$. Let $k > 0$ be a positive integer and denote the set of the first $2k$ integers by $[2k] = \{1, \ldots, 2k\}$. Let $A \subset [2k]$ be a subset of the first $2k$ integers and let

$$\delta(A, \varepsilon, 2k) = \left| \sum_{i \in A} t_i(\varepsilon) - \sum_{i \in [2k] \setminus A} t_i(\varepsilon) \right| \ .$$

Define

$$\delta(\varepsilon, 2k) = \min_{A \subset [2k]} \delta(A, \varepsilon, 2k) \ .$$

Question 4. *What is the value of $\delta(\varepsilon, 2k)$?*

Question 5. *If $n = 2k+1$ is an odd integer then how small is the value of $\delta(\frac{4}{n^2}, n-1)$?*

If there is an answer to Question 5 that it is smaller then some exponential function of n (like e^{-cn}) then there would be a triangulation of the unit square into n triangles such that the area of each triangle is exponentially close to $\frac{1}{n}$.

On order of degree matrices

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We say that a square matrix $\mathbf{M}$ of order r is a degree matrix of a given graph G if there is a so called equitable partition of its vertices into r blocks with the following property: For any i and j it holds that a vertex from the i-th block of the partition has exactly $m_{i,j}$ neighbors inside the j-th block.

For two degree matrices M of order k and N of order l we write $M \to N$ if there exists a partition $R_1, \ldots, R_l$ of the set $\{1, 2, \ldots, k\}$ such that:

If $i, i' \in R_r$ then for all $s = 1, \ldots, l$ holds

$$\sum_{j \in R_s} m_{ij} = \sum_{j \in R_s} m_{i'j} = n_{rs}$$

The comparison $M \to N$ means that given an graph G with an equitable partition $\mathcal{B} = \{B_1, \ldots, B_k\}$ for the matrix $\mathbf{M}$ we can form a partition $\mathcal{C} = \{C_1, \ldots, C_k\}$ with matrix $\mathbf{N}$ only by joining some classes, in particular by

$$\forall j = 1, \ldots, l: \; C_j = \bigcup_{i \in R_j} B_i.$$

(In such a case $\mathcal{C}$ would be coarser than $\mathcal{B}$ and we would write $\mathcal{C} \preceq \mathcal{B}$.)

Example:

$$M = \begin{pmatrix} 1 & 1 & 2 & 0 \\ 1 & 0 & 2 & 3 \\ 1 & 2 & 0 & 3 \\ 0 & 1 & 1 & 2 \end{pmatrix} \xrightarrow[\{1\},\{2,3\},\{4\}]{R_1,R_2,R_3=} \begin{pmatrix} 1 & 3 & 0 \\ 1 & 2 & 3 \\ 0 & 2 & 2 \end{pmatrix} = N$$

It is known [2] that the order $\to$ is a disjoint union of partial orders and each such suborder has a minimal element — a special degree matrix of the coarsest equitable partition (up to a permutation of rows an columns).

Problems and solutions

1. It is known that equitable partitions of a single graph form a lattice under the partial order $\preceq$ [1]. The question is whether this translates to the partial order $\to$ on the corresponding degree matrices.

   Solution: The answer is negative, consider e.g. an example by Jan Kratochvíl Take the following matrices (corresponding to partitions of K_5):

   $$\mathbf{M}_1 = \begin{pmatrix} 2 & 1 & 1 \\ 3 & 0 & 1 \\ 3 & 1 & 0 \end{pmatrix} \quad \mathbf{M}_2 \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 2 & 2 & 0 \end{pmatrix}$$

   $$\mathbf{N}_1 = \begin{pmatrix} 2 & 2 \\ 3 & 1 \end{pmatrix} \quad \mathbf{N}_2 = \begin{pmatrix} 3 & 1 \\ 4 & 0 \end{pmatrix}$$

   and compare them by: $M_1 \xrightarrow{\{1\},\{2,3\}} N_1$, $M_1 \xrightarrow{\{1,2\},\{3\}} N_2$, $M_2 \xrightarrow{\{1,3\},\{2\}} N_1$, $M_2 \xrightarrow{\{1,2\},\{3\}} N_2$,

2. By a theorem of Leighton [3] it is well known that for any two graphs G_1 and G_2 such that their coarsest equitable partitions give the same degree matrix there exists a graph G covers (i.e. allows a locally

bijective homomorphism) to both G_1 and G_2. Consequently for any two degree matrices $\mathbf{M}_1$, $\mathbf{M}_2$ from the same suborder one can find a matrix $\mathbf{M}$ that is above both of them, i.e. $\mathbf{M} \to \mathbf{M}_1$ and $\mathbf{M} \to \mathbf{M}_2$. (Just for $i = 1, 2$ find some G_i with degree matrix $\mathbf{M}_i$ and take $\mathbf{M} = \mathrm{Adj}(G)$.)

The question is whether one can establish a direct construction of $\mathbf{M}$ from $\mathbf{M}_1$, $\mathbf{M}_2$ without use of the Leighton's theorem.

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Matchings in hypercubes

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The n-dimensional hypercube Q_d is a graph whose vertex set consists of all binary vectors of length d, two vertices being adjacent whenever the corresponding vectors differ in exactly one coordinate. Let M be an arbitrary perfect matching in Q_d.

Kreweras's conjecture: There exists a Hamiltonian cycle C in Q_d containing M for every perfect matching M where $d \geq 2$.

Let us consider a Hamiltonian cycle C containing a perfect matching M. Then $C \setminus M$ is a perfect matching in Q_d. Hence we need to find a perfect matching N in $E(Q_d) \setminus M$ such that $N \cup M$ is a Hamiltonian cycle. We know that there exists a perfect matching in $E(Q_d) \setminus M$ because this subgraph is $(d-1)$-regular and bipartite. But $M \cap N$ is several independent cycles for most perfect matching N in the subgraph.

The conjecture holds for $d \leq 5$. The tree dimensional hypercube has only two perfect matchings up to isomorfism and it is easy to find the Hamiltonian cycle for both perfect matchings. In four dimensions we found all perfect matchings and corresponding Hamiltonian cycles. We used a computer to verify the conjecture for five dimensional hypercube. Higher dimension is too difficult even for computers.

Tom J. Rackham shows following implication in HOMONOLO. If the conjecture holds for $d = k - 1$ and perfect matching M in Q_k do not use one dimension (i.e. we can divide Q_k into two $k-1$ dimensional subcubics such that every edge $e \in M$ belongs into one of those subcubics) than there exists a Hamiltonian cycle in Q_k containing M.

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Häggkvist and Hell's K_3-mote universal graph

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Abstract

We consider a construction of Häggkvist and Hell that provides a triangle free graph U which admits homomorphisms from any triangle free graph with maximum degree at most 3.

We ask if this is the smallest such graph, and we ask if U-colouring is NP-complete for graphs of maximum degree 4.

In [1], Häggkvist and Hell construct for any graph A and any integer b a graph $U = U(A, b)$ such that $A \not\to U$, (i.e. U is A-mote), but that for any A-mote graph G of maximum degree b, $G \to U$. Thus for graphs G of bounded degree b, this gives the duality

$$A \not\to G \iff G \to U(A, b).$$

Homonolo 2002 conference photos – in the misty weather at the nature preserve "Velká Klečová Louka" bog, and in front of Šámalova chata. The cold weather calls for good gloves, and we had them.

Active sport life during the sunny afternoons of Homonolo 2003.

(2003) When biking at speed of 40 mph in the rocky terrain does not provide enough adrenalin, substitute activities are looked for by some participants.

The beautiful Indian summer of Homonolo 2003 invited us to cool down in the Černá Nisa reservoir. And some participants felt the temptation to strip into swimming trunks even for the conference photo in front of the lodge.

The winter of 2005 started during the Homonolo week. And offered us unusual hiking experiences.

(2006) The cozy interier of Šámalova chata provides a unique workshop atmosphere. In the bottom picture, the hikers enjoy a rare glimpse of sunshine, an hour before a blizzard came.

(2006) The rocky formations “Sněžné věžičky” are a popular target of afternoon hikes.

Homonolo 2006 conference photo. Left to right: Robert Babilon, Eva Ondráčková, Jiří Fink, Tom Rackham, Attila Pór, Ondřej Suchý, Ivan Dovica, Vít Jelínek, Mark H. Siggers, Dirk Schlatter, Maya J. Stein.

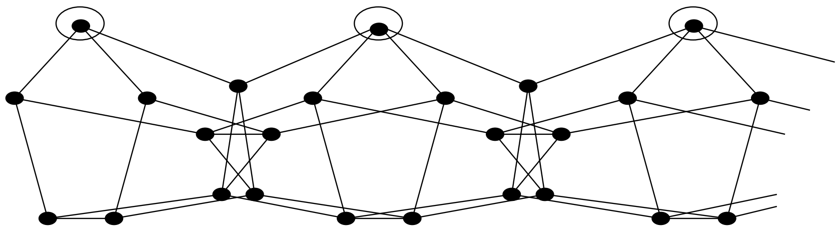

Figure 1:

Since for fixed A the problem of existence of a homomorphism $A \to G$ is solvable in time polynomial in the size of G, this implies that the U-colouring problem is also polynomial time solvable for instances of bounded degree 3.

Specializing to the case where $b = 3$ and $A = K_3$ the construction of $U = U(K_3, 3)$ is as follows.

Construction 1. *Let $S = \{1, \ldots, 22\}$ be a set of labels, and let*

$$V(U) = \{(a, \{b, c, d\}) \mid a, b, c, d \text{ are distinct elements of } S\}.$$

Define U by making two vertices $(a, \{b, c, d\})$ and $(a', \{b', c', d'\})$ adjacent if

i. $a \in \{b', c', d'\}$,

ii. $a' \in \{b, c, d\}$, *and*

iii. $\{b, c, d\} \cap \{b', c', d'\} = \emptyset$.

For a vertex $(a, \{b, c, d\})$ we refer to a as its *center label* and to b, c, and d as its *neighbour labels*. It will be convenient to talk of a U-colouring of a graph G as a labelling of its vertices in which each vertex is labelled with a center, and three neighbour labels, and adjacent vertices get labelled in accordance with the conditions above.

Observing that in a U-colouring of C_5, ever vertex of C_5 must get a distinct center label, one can quickly show that the Petersen graph P has a unique U-colouring up to permutation of the labels. And indeed, in the graph in Figure 1, constructed from copies of P, the circled vertices get the same labels under any U-colouring.

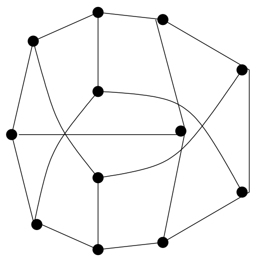

Figure 2:

Using this gadget, and the fact that U is vertex transitive, one can construct for and K_3-free graph G, a K_3-free graph G' of maximum degree 5 such that

$$G' \to U \iff G \to U.$$

Since U is non-bipartite, and so U-colouring is NP-complete, this implies that U-colouring is NP-complete for graphs of maximum degree at most 5. Strengthening this, one would like to decide the following.

Question 6. *Is $U(K_3, 3)$-colouring NP-complete for instances with maximum degree at most 4?*

The proof in [1] that any K_3-free 3-bounded graph G yields $G \to U$ uses the fact that such a G has a 22-colouring in which no two vertices of distance at most 3 apart have the same colour. With Attila Por, we observed that the proof holds if G has a colouring in which no pair of vertices of distance 1 or 3 get the same colour. This can be done with 16 colours. Thus proof holds if S in the construction of $U(K_3, 3)$ is replaces with a set of 16 labels. With a bit more work we showed that even 15 labels is enough. On the other hand, finding a cubic graph on 12 vertices such that there is a path of length 1 or 3 between every pair of vertices showed us that the S could not be replaced by a set of 11 labels. (See Figure 2.)

This raised the following question, where $U(K_3, 3, n)$ is the construction from [1] with the set of labels S replaced with a set of n labels.

Question 7. *What is the minimum n such that $U = U(K_3, 3, n)$ yields $G \to U$ for any K_3-free graph G of maximum degree 3.*

We can show that $12 \le n \le 15$.

In general, the construction $U(A, b)$ can be analogously refined to $U(A, b, n)$ by replacing some set of labels with a smaller set of n labels. For each A and b we would like to find $n(A, b)$, the minimum n for which $U = U(A, b, n)$ admits $G \to U$ for all A-mote graphs G with maximum degree b. We would then like to show that $U(A, b, n(A, b))$-colouring is NP-complete form graphs of maximum degree $b + 1$.

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Deciding tree duality

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Let $G = (V, E)$ and $H = (W, F)$ be two digraphs. A *homomorphism* from G to H is a mapping $f : V \to W$ such that for every edge (u, v) of G, the pair $\big(f(u), f(v)\big)$ is an edge of H. See [4] for more on graph homomorphisms.

For a fixed digraph H (a *template*), the *H-colouring problem* is the decision problem whether for an input digraph G there exists a homomorphism from G to H. The complexity of H-colouring depends on the digraph H, and it was conjectured in [2], that for any H it is either polynomial or NP-complete. This is known when H is symmetric ([3]), but in general the problem remains open.

For several templates H there exists a polynomial algorithm due to so-called duality characterisations (good characterisations in the sense of [1]). For example, if there exists a digraph F such that there is a homomorphism from an input G if and only if there does *not* exist a homomorphism from F to G, then H-colouring is tractable: the exhaustive-search algorithm that checks all possible mappings from $V(F)$ to $V(G)$ decides the problem in time $O(|G|^{|F|})$, polynomial in the size of the input G. The duality condition above can be stated in a short form:

$$F \not\to = \to H.$$

We say that H *has tree duality* if for any digraph G there exists no homomorphism from G to H if and only if there exists a tree T such that T admits a homomorphism to G but not to H, in other words:

$$\mathcal{T}_H \not\to = \to H,$$

where $\mathcal{T}_H$ is the class of all trees that do not admit a homomorphism to H. (A tree in digraphs is simply an orientation of a tree.)

It was proved in [5] that H-colouring is tractable if H has tree duality.

We are interested in the complexity of the "metaquestion": *how hard is to decide whether an input digraph has tree duality?*

This is known to be decidable by [2]: For a digraph H, the *power graph* of H is the digraph $\mathcal{P}(H)$ with vertex set $\mathcal{P}(V)$ and with $(W, W') \in E(\mathcal{P}(H))$ if and only if for each $w \in W$ there exists $w' \in W'$ such that (w, w') is an edge of H and for each $w' \in W'$ there exists $w \in W$ such that $(w, w') \in E(H)$. Then H has tree duality if and only if there exists a homomorphism from $\mathcal{P}(H)$ to H.

However, the size of the power graph is exponential in the size of H, so the characterisation above does not even prove that tree duality is in NP.

It may be easier to solve the problem at least for inputs restricted to trees; then some useful properties of the power graph may be found and it might be possible to simplify the decision procedure.

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Spanning trees in infinite graphs

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We present two problems on spanning trees in infinite graphs.

The first problem is probably more accessible. As in finite graphs, define a *spanning tree* of an infinite graph as a connected spanning subgraph that contains no (finite) cycles, i.e. that has no finite 2-regular subgraphs. Diestel asked:

Problem 8. *Is there a function $f : \mathbb{N} \to \mathbb{N}$ so that any $f(k)$-edge-connected graph contains k edge-disjoint spanning trees?*

For finite graphs, the answer follows from Nash-Williams' and Tutte's tree-packing theorem[2]: $f(k) = 2k$ will do. Indeed, it is easily seen that in a $2k$-edge-connected graph G, for any partition of the vertex set into p sets, the number of cross-edges is at least $k(p-1)$. Thus the tree-packing theorem implies that G has k edge-disjoint spanning trees.

In infinite graphs, however, the tree-packing theorem does not literally extend, and therefore cannot be used to prove a statement as in Problem 1. It is known, and not difficult to prove (see [2]), that every connected graph has a spanning tree. This settles the problem for $k = 1$, but for $k \geq 2$, the question is unanswered yet.

[2]This well-known theorem states the equivalence of the following two properties for a finite graph G:

- for any partition of $V(G)$ into p sets, G has at least $k(p-1)$ cross-edges (edges between partition classes), and
- G has k edge-disjoint spanning trees.

Recent research, however, suggests that, seeing the problem from a different perspective, we might consider *topological spanning trees* instead of 'classical' spanning trees. With these, a literal extension of the tree-packing theorem is indeed possible, and thus the answer to Problem 1 follows as in the finite case. For details on infinite topological tree-packing, see [4], or [2].

Topological spanning trees have been introduced alongside *infinite cycles* by Diestel and Kühn [3]. For an easy introduction to this topic, and an overview of the applications of their theory, see [1], and [2]. We shall give only a brief definition of the concepts needed here, nevertheless we need some standard infinite graph-theoretic background.

A one-way infinite path is called a *ray*, and an *end* of a graph is an equivalence class of rays, where two rays are equivalent if no finite set of vertices separates them. In a very natural way, one can define a topology on the point set of a graph G plus its ends (thus a point may be a vertex, an end, or an inner point of an edge), so that the vertices of any ray converge to the end to which it belongs.

Begin by endowing G itself with the usual topology of a 1-complex. That is, every edge is a homeomorph of the interval $[0, 1]$, the basic open neighbourhoods of an inner point z of an edge e are simply the open 'subintervals' of e that contain z, and the basic open neighbourhoods B_ε of a vertex v are the unions of half-open intervals $[v, z)$, one for every edge e at v, with z the image of ε under the homeomorphism from $[0, 1]$ to e.

For a given end ω of G, define for each finite set $S \subseteq V(G)$, and each $\varepsilon \in (0, 1)$, an open neighbourhood $B_{S,\varepsilon}$ of ω in the following way. Let $C(S, \omega)$ be the (unique) component of $G - S$ in which every ray from ω has a tail. Add to $C(S, \omega)$ the set of all ends whose rays have a tail in $C(S, \omega)$, and also add the half-edges $[v, z)$, one for every S–$C(S, \omega)$ edge $e = vw$ of G, with z the image of ε under the homeomorphism from $[0, 1]$ to e. This gives $B_{S,\varepsilon}$.

A *cycle* in the thus obtained topological space $|G|$ is the homeomorphic image of the unit circle in $|G|$, or, more precisely, the set of edges corresponding to a such. This definition includes traditional finite cycles, as well as many more complicated ones. It is now possible to define an infinite analogue of the *cycle space* of a finite graph. See [1] for details.

Finally, define a *topological spanning tree* T of a graph G as an arc-connected subset of $|G|$ that contains each edge of which it contains an inner point, and that contains all vertices of G, but no cycle. Then T is clearly closed, and since T contains $V(G)$, it also contains all ends of G.

Now, the subgraph corresponding to a topological spanning tree need not be a 'classical' spanning tree: it might simply lack the (graph-theoretic) connectedness. For example, consider the so-called *infinite ladder* L on the vertex set $\bigcup_{i\in\mathbb{N}}\{v_i, w_i\}$ with edges v_iv_{i+1}, w_iw_{i+1}, and v_iw_i for all $i \in \mathbb{N}$. Delete all edges of the form v_iw_i, and we obtain a subgraph H whose closure in $|L|$ is a topological spanning tree. But H is not connected.

Neither need the closure of a 'classical' spanning tree be a topological spanning tree. There can be two reasons for this: on one hand, there might turn up infinite cycles (as in the example above, if we add the edge v_4w_4 to H), on the other hand, the closure might fail to be arc-connected. This latter problem may occur only in graphs with infinite degrees. As an example may serve a ray whose vertices are identified with the leaves of an infinite star. The star is a spanning tree whose closure is not arc-connected.

So one problem with topological spanning trees is whether they exist:

Problem 9. [**3**] *Does every graph have a topological spanning tree?*

So far this is only known for special classes of graphs, in particular for those that have a *normal* spanning tree (defined as for finite graphs). This includes all countable connected graphs.

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The pinning number of axis-parallel rectangles

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Assume that we are given a collection $\mathcal{C}$ of axis-parallel rectangles in the plane (a rectangle is called axis-parallel, if its edges are horizontal and vertical segments). The *independence number of* $\mathcal{C}$ (denoted by $\alpha(\mathcal{C})$) is the maximum size of a subset of $\mathcal{C}$ formed by pairwise disjoint rectangles. The *pinning number of* $\mathcal{C}$ (denoted by $\tau(\mathcal{C})$) is the minimum size of a set X of points in the plane ("pins") with the property that every rectangle from $\mathcal{C}$ contains at least one point from the set X. Let us define $\tau(n)$ to be the maximum of $\tau(\mathcal{C})$ over all collections $\mathcal{C}$ of axis-parallel rectangles with $\alpha(\mathcal{C}) = n$.

It is not hard to see that $\tau(1) = 1$; in other words, every family of pairwise intersecting axis-parallel rectangles has nonempty intersection. It is also known that $\tau(2) = 3$ and $\tau(3) = 5$. However, to establish the asymptotics of $\tau(n)$ as n grows to infinity is a long-standing open problem. Clearly, we have the inequality $\tau(n) \geq n$, because n disjoint rectangles must be pinned by n distinct pins. In fact, it is possible to prove $\tau(n) \geq 2n - O(1)$. There are also several different proofs of the upper bound $\tau(n) \leq O(n \log n)$ (see [1,2]). However, so far nobody has been able to obtain more precise bounds, even though it was conjectured long ago in [3] that $\tau(n) < 2n$.

Let us remark that this problem can be formulated entirely in the language of intersection graphs. For a family $\mathcal{C}$ of axis-parallel rectangles, we define the *intersection graph* to be the graph G whose vertices are the rectangles of $\mathcal{C}$ and whose edges correspond to the pairs of rectangles with nonempty intersection. In this terminology, the independence number of $\mathcal{C}$ is exactly the independence number of G and the pinning number of $\mathcal{C}$ is the chromatic number of the complement of G.

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Decreasing graph properties

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Consider a decreasing graph property $\mathcal{P}$, i.e. a set of graphs which is closed under isomorphism and deletion of edges, and denote the subset of graphs on $[n] := \{1, \ldots, n\}$ by $\mathcal{P}_n$ We say that $\mathcal{P}$ is

(i) *addable* if for every $P \in \mathcal{P}$ and vertices u and v lying in different components of P, the graph $P + uv$ also lies in $\mathcal{P}$,

(ii) *small* if there exists a constant $\gamma > 0$ such that $|\mathcal{P}_n| \leq n!\gamma^n$, and

(iii) *sparse* if there exists a constant $\lambda > 1$ such that every graph in $\mathcal{P}_n$ has less than λn edges.

Question 10. *Does there exist a decreasing graph property which is addable and small, but not sparse?*

Vít Jelínek has constructed a suitable decreasing graph property which shows that the answer is affirmative. Setting $\omega(n) := (\log n)^{1/3}$, it may be roughly described as the family of all copies of subgraphs of $K_{\omega(n),n-\omega(n)}$ and possibly some bridges (for addability). For a formal definition, denote the copies of $K_{\omega(n),n-\omega(n)}$ on $[n]$ by $\mathcal{B}_n$ and set

$$\mathcal{P}_n := \{G \in \mathcal{G}_n \,|\, \exists B \in \mathcal{B}_n \,\forall e \in E(G)\,(e \in E(B) \,\vee\, e \text{ is a bridge of } G)\}\,.$$

This $\mathcal{P}$ is obviously an addable decreasing graph property and, as any element of $\mathcal{B}_n$ has $\sim \omega(n) \cdot n$ edges and is contained in $\mathcal{P}_n$, $\mathcal{P}$ is not sparse. To get an upper bound for $|\mathcal{P}_n|$, fix some $B_0 \in \mathcal{B}_n$ and consider $\mathcal{P}_n(B_0) := \{G \in \mathcal{G}_n \,|\, \forall e \in E(G)\,(e \in E(B_0) \,\vee\, e \text{ is a bridge of } G)\}$. Let U and V be the sets which partition $V(B_0)$ into independent sets of size $\omega(n)$

and $n - \omega(n)$, respectively. For an arbitrary $G \in \mathcal{P}_n(B_0)$ we colour the edges in $E(G) \cap E(B_0)$ *blue* and the edges in $E(G) \setminus E(B_0)$ *red* and denote set of vertices in V which are incident to at least one blue edge by $W(G)$.

We can then split $\mathcal{P}_n(B_0)$ into

$$\mathcal{P}'_n(B_0) := \{G \in \mathcal{P}_n(B_0) \mid |W(G)| \leq n/\omega(n)\}$$

and

$$\mathcal{P}''_n(B_0) := \{G \in \mathcal{P}_n(B_0) \mid |W(G)| > n/\omega(n)\}.$$

To obtain an upper bound for $|\mathcal{P}'_n(B_0)|$ we construct elements of $\mathcal{P}'_n(B_0)$ as follows. Start with the empty graph on $U \cup V$ and select $n/\omega(n)$ vertices in V. Then connect each of these vertices arbitrarily to vertices in U. In the final step, add a set of bridges not in $E(B_0)$ to the graph.

Clearly, for each element of $\mathcal{P}'_n(B_0)$ there is at least one way to construct it in this way. Since the edges added in the last step cannot contain (the edges of) a cycle, we get

$$\begin{aligned} |\mathcal{P}'_n(B_0)| &< \binom{n-\omega(n)}{n/\omega(n)} \cdot \left(2^{\omega(n)}\right)^{n/\omega(n)} \cdot |\mathcal{F}_n| \\ &\leq (e\omega(n))^{n/\omega(n)} \cdot 2^n \cdot e^{n+o(n)} n! \\ &= (2e)^{n+o(n)} n!. \end{aligned} \tag{1}$$

To construct elements of $\mathcal{P}''_n(B_0)$ we likewise first insert the blue edges and then the red ones. Suppose we have inserted the blue edges in a certain appropriate way, i.e. for some edge set $E_b \subseteq E(B_0)$ the graph $G_1 := (U \cup W, E_b)$ has more than $n/\omega(n)$ non-isolated vertices in V. As each such vertex $v \in W(G_1)$ is connected to at least one vertex in U, the graph $G_1[U \cup W_{G_1}]$ has at most $|U| = \omega(n)$ components. This implies in particular that G_1 has a component C of size larger than $n/\omega(n)^2$. If T is a spanning tree of C and E_r a set of bridges (not in $E(B_0)$) which we might insert in the second step, then $(U \cup V, E(T) \cup E_r)$ is a forest. Moreover, each different choice of T and E_r will give rise to a different element of $\mathcal{F}_n$, and thus

$$\left|\left\{ E_r \subseteq [n]^{(2)} \setminus E(B_0) \,\middle|\, G_1 + E_r \in \mathcal{P}''_n(B_0)\right\}\right| \cdot \left|\mathcal{F}_{n/\omega(n)^2}\right| < |\mathcal{F}_n|,$$

giving

$$
\begin{aligned}
|\mathcal{P}''_n(B_0)| &< 2^{\omega(n)\cdot(n-\omega(n))} \cdot \frac{|\mathcal{F}_n|}{\left|\mathcal{F}_{n/\omega(n)^2}\right|} \\
&= 2^{\omega(n)\cdot n} \cdot \frac{e^{n+o(n)} n!}{e^{\frac{n}{\omega(n)^2}+o\left(\frac{n}{\omega(n)^2}\right)} \left(\frac{n}{\omega(n)^2}\right)!} \\
&= 2^{\omega(n)\cdot n} \frac{e^{n+o(n)} n!}{\left(\frac{n}{\omega(n)^2}\right)^{\frac{n}{\omega(n)^2}}} \\
&= e^{n+o(n)} n! \cdot \exp\left(n\left(\omega(n)\log 2 - \frac{\log n}{\omega(n)^2} + \frac{\log(\omega(n)^2)}{\omega(n)^2}\right)\right) \\
&\ll e^n n!, \qquad (2)
\end{aligned}
$$

by our choice of $\omega(n)$. Combining (1) and (2) we finally get

$$
\begin{aligned}
|\mathcal{P}_n| &\leq \sum_{B\in\mathcal{B}_n} |\mathcal{P}'_n(B_0)| + |\mathcal{P}''_n(B_0)| \\
&\leq \binom{n}{\omega(n)} (2e)^{n+o(n)} n! + e^n n! \\
&= (2e)^{n+o(n)} n!.
\end{aligned}
$$

Minimal product of $\chi(G)$ and $f(G)$

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Origin: Alex D. Scott.

Let $f(G)$ be the size of the largest induced subgraph of G with every degree odd.

Conjecture 11. *$\chi(G)f(G) \geq n$ if $n = |V(G)|$.*

There was a bipartite counterexample of nine points with $f(G) = 4$. For any odd number n we show a bipartite counterexample of size n^2 with $f(G) = \frac{n^2-1}{2}$. As a new conjecture one may ask whether $\chi(G)f(G) \geq n-1$?

Let n be an odd integer and let $A = [n] = \{1, \ldots, n\}$ be the set of the first n integers. Let $B = \{b(x,y) \mid x, y \in [n], x < y\}$ and $C = \{c(x,y) \mid x, y \in [n], x < y\}$. Let G be a bipartite graph on the vertices $V(G) = A \cup B \cup C$ where A will be one color class and $B \cup C$ the other color class. The edges of G are $(x, b(x,y)), (y, b(x,y)), (x, c(x,y)), (y, c(x,y)) \in E(G)$ for any $x, y \in A$.

Let $X \subset V(G)$ be a subset of the vertices such that the induced graph $G(X)$ on X has all degrees odd. Let $X_A = X \cap A$ and $X' = X \cap (B \cup C)$. Since every vertex in $B \cup C$ has degree two therefore they must be leafs in $G(X)$. Let $t = |X_A|$. If $b(x,y) \in X$ then X_A contains exactly one of the two elements x and y. The same is true for $c(x,y)$ and therefore the degree of $x \in X_A$ in $G(X)$ is at most $2(n-t)$. Since its degree $\deg_X(x)$ is odd it can be at most $2(n-t)-1$. Since all vertices in $G(X)$ that are in $B \cup C$ are leafs we have

$$\begin{aligned} |V(G(X))| &= t + \sum_{x \in X_A} \deg_X(x) \leq t + t(2(n-t)-1) = 2t(n-t) \\ &\leq 2\frac{n-1}{2}\frac{n+1}{2} = \frac{n^2-1}{2} \quad . \end{aligned}$$

This shows that $f(G) \leq \frac{n^2-1}{2}$ but it is easy to come up with an example such that $f(G) \geq \frac{n^2-1}{2}$.

NP-completeness of switching to H-free graphs

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Let G be a graph. Then the *Seidel's switch of a vertex subset* $A \subseteq V_G$ is called $S(G, A)$ and

$$S(G,A) = (V_G, E_G \triangle \{xy : x \in A,\ y \in V_G \setminus A\}),$$

where $\triangle$ denotes the symmetric difference of sets. We say that two graphs G and H are *switching equivalent* (denoted by $G \sim H$) if there is a set $A \subseteq V_G$ such that $S(G, A)$ is isomorphic to H. The set

$$[G] = \{S(G, A) : A \subseteq V_G\}$$

is called the *switching class* of G.

We call a graph H-free if it does not contain H as an induced subgraph. For a fixed graph H, we consider the following decision problem:

> **Input:** A graph G
> **Question:** Does there exist a set $A \subseteq V_G$ such that $S(G, A)$ is H-free?

We study the computational complexity of the problem depending on the fixed graph H. It is easy to see that if an algorithm $\mathcal{A}$ decides whether G is switching-equivalent to an H-free graph, then $\mathcal{A}$ also decides if $\overline{G}$ is switching-equivalent to an $\overline{H}$-free graph. Hence we can identify the results for H and $\overline{H}$ in the following text.

Hage et al. [1] proved that a graph is switching-equivalent to a K_2-free graph if and only if it does not contain an induced K_3 nor $K_2 + K_1$. Thus, switching-equivalence to a K_2-free graph can be decided in time $O(n^3)$ by checking all three-vertex induced subgraphs of the input graph.

Kratochvíl et al. [4] described an algorithm which runs in time $O(n^3)$ and decides if a given graph can be switched not to contain an induced $K_{1,2}$.

Hayward [2] found an $O(n^3)$-time algorithm for P_3-structure recognition, using a connection between P_3-structures and switching classes. A part of this algorithm is a procedure for switching a graph to an I_3-free graph. Independently of this result, Hage et al. [1] also found an $O(n^3)$-time algorithm for switching to a K_3-free graph.

The case of P_4 has been solved by Hertz [3]. Hertz characterized the graphs that are switching-equivalent to a P_4-free graph by four forbidden induced subgraphs, each having five vertices. This characterization immediately gives an $O(n^5)$-time recognition algorithm.

Ondráčková [5] has proved that switching-equivalence to a $K_{1,3}$-free graph can be decided in time $O(n^6)$. Thus, all known results for particular graphs H yield polynomial-time algorithms, and the following problem remains open.

Problem 12. *Is there a graph H such that the problem of deciding switching-equivalence to an H-free graph is NP-complete?*

Characterizations by forbidden induced subgraphs

For certain graphs H, slightly more than the polynomial-time decision algorithm is known. There exists a characterization of graphs switching-equivalent to and H-free graph by a finite set of forbidden induced subgraphs, i.e., a set $\{F_1, \ldots, F_k\}$ such that a graph G is switching-equivalent to an H-free graph if and only if $F_1, \ldots, F_k \not\leq G$.

The set of forbidden induced subgraphs exists for any H, but it does not have to be finite. The sets are known for K_2 (as mentioned above), for P_4 due to Hertz [3], and for $K_{1,2}$ due to Ondráčková [5]. In all these cases, the sets are finite and small, which yields the following problem.

Problem 13. *Is there a graph H such that the set of minimal forbidden induced subgraphs for the property "being switching-equivalent to an H-free graph" is infinite?*

Of course, if there is a graph H such that deciding switching-equivalence to an H-free graph is NP-complete, then the list of minimal forbidden induced subgraphs is surely infinite (assuming that P is not NP). So a positive answer to Problem 12 is also a positive answer to Problem 13, but not conversely.

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